MULTIVARIATE NONPARAMETRIC TESTS FOR INDEPENDENCE AND FOR MULTI-SAMPLE LOCATION PROBLEM

By

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To my parents, my wife

and

my children, Yeun and Yechan

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KEY TO SYMBOLS

Symbol	Definition
$\boldsymbol{a} = (a_1, \ldots, a_s)'$	Vector
$\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_t) = \{a_{ij}\}_{s \times t}$	Matrix
$\boldsymbol{A}' = \{a_{ji}\}_{t \times s}$	Transpose of $\boldsymbol{A}$
$\lvert\boldsymbol{A}\rvert$	Determinant of $\boldsymbol{A}$
$\mathrm{tr}(\boldsymbol{A})$	Trace of $\boldsymbol{A}$
$\mathrm{vec}\,(\boldsymbol{A}) = (\boldsymbol{a}_1', \ldots, \boldsymbol{a}_t')'$	Vector of $\boldsymbol{A}$
$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & \ddots \end{pmatrix}$	Direct product
$\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$	Direct sum
$\mathbb{R}$	Real numbers
Ω_p	Unit hypersphere of dimension p
∇	Gradient operator
$\sim$	Distributed as
iid	Independent and identically distributed
cdf	Cumulutave distribution function

Symbol/Definition	Term
$\mathrm{E}[\cdot]$	Expectation
$\mathrm{Cov}[\cdot,\cdot]$	Covariance
$\mathrm{V}[\cdot]$	Variance
$\xrightarrow{d}$	Converge in distribution
$\xrightarrow{p}$	Converge in probability
$AN(\mu,\sigma^2)$	Asymptotically normal r.v. with mean μ and variance σ^2
$\chi_k^2(\lambda)$	Chi-square r.v. with k d.f. and noncentrality parameter λ
ARE	Asymptotic relative efficiency
$o_p(f(n))$	Term when divided by $f(n)$ converges to zero in probability as $n \to \infty$
$O_p(f(n))$	Term when divided by $f(n)$ is bounded in probability as $n \to \infty$

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A multivariate nonparametric statistic based on interdirections is proposed for testing the independence among many vectors. This statistic is an extension of the interdirection quadrant statistic introduced by Geiser (1993) for the case of two vectors. The proposed statistic is affine-invariant under a certain class of nonsingular linear transformations and has an asymptotic chi-square distribution under the null hypothesis of independence when each vector is elliptically symmetric. Comparisons are made among the proposed statistic, Wilks likelihood ratio criterion, and a component-wise quadrant statistic via Pitman asymptotic relative efficiencies (ARE's) and Monte Carlo studies. The Pitman ARE's indicate that the proposed statistic performs better than Wilks likelihood ratio criterion for heavy-tailed distributions and is also better than the component-wise quadrant statistic for a very broad class of underlying distributions. Monte Carlo results show that the proposed statistic performs better than others when the underlying distributions are heavy-tailed.

Also nonparametric tests for the multi-sample multivariate location problem are proposed which extend the two-sample multivariate rank tests by Randles and Peters

(1990) to the multi-sample setting. The asymptotic distributions of the proposed statistics under the null hypothesis and under certain contiguous alternatives are obtained for a class of elliptically symmetric distributions. Comparisons are made between the proposed statistics and several competitors via Pitman asymptotic relative efficiencies and Monte Carlo results. The tests proposed perform better than the Lawley-Hotelling's generalized T^2 for heavy-tailed distributions. For normal to light-tailed distributions, the proposed statistics also dominate the other competitors and the proposed analog of signed-rank test performs better than the Lawley-Hotelling's generalized T^2 for light-tailed distributions.

CHAPTER 1
INTRODUCTION

This dissertation concerns two distinct problem settings. The first setting involves testing of independence among a set of vectors using data that consist of multiple observations of the sets of vectors. The second problem setting is that of the multivariate, many sample location problem in which the objective is testing whether several multivariate populations have a common location (or different locations) under an unrestricted shift in location model. These two problem settings are common in that the number of vectors and samples are generalized to many vectors and samples respectively.

For the independence problem setting, we let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ denote a random sample of vectors from a continuous distribution with $\boldsymbol{X}_i = (\boldsymbol{X}_i^{(1)'}, \boldsymbol{X}_i^{(2)'}, \ldots, \boldsymbol{X}_i^{(c)'})'$, where $\boldsymbol{X}_i^{(\alpha)}$ is a $r_\alpha \times 1$ and has a marginal density $f_\alpha\left(\boldsymbol{x}^{(\alpha)}\right)$, $\alpha = 1, 2, \ldots, c$. Note that the dimensions of the vectors need not be the same. We are interested in testing

$$H_0\colon\, f_X\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(c)}\right) = f_1\left(\boldsymbol{x}^{(1)}\right) f_2\left(\boldsymbol{x}^{(2)}\right) \cdots f_c\left(\boldsymbol{x}^{(c)}\right)$$

versus

$$H_1\colon\, f_X\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(c)}\right) \neq f_1\left(\boldsymbol{x}^{(1)}\right) f_2\left(\boldsymbol{x}^{(2)}\right) \cdots f_c\left(\boldsymbol{x}^{(c)}\right).$$

Thus under H_0 the component vectors in $\boldsymbol{X}_i$ are independent while under the alternative they exhibit some type of dependence. We first consider the bivariate test of independence between two univariate random variables $X^{(1)}$and $X^{(2)}$ as a special case of this problem, that is, c = 2 and the value of each r_α is 1. Over the past century there has been much research on the correlation between two univariate random

variables and many bivariate measures of correlation have been proposed. Probably the most well-known test procedure is based on the Pearson product moment correlation coefficient (r) (Pearson 1896). This test is particularly effective if the underlying distribution is bivariate normal. Numerous nonparametric approaches to this problem have been proposed based on the ranks of $X_i^{(1)}$'s ($X_i^{(2)}$'s) denoted by $R_1, \ldots, R_n$ ($Q_1, \ldots, Q_n$). Examples include Spearman's rho (ρ) (Spearman 1904) Kendall's tau (τ) (Greiner, 1909; Kendall, 1938) and the quadrant statisric (q') (Blomqvist, 1950). Comparisons among the tests of independence based on these rank correlation coefficients have been made using asymptotic relative efficiencies for various models of dependence.

Konijin (1956) used the model of dependence $X^{(1)} = \alpha_1 U + \alpha_2 V$ and $X^{(2)} = \alpha_3 U + \alpha_4 V$ where U and V are independent and α_1,α_2,α_3, and α_4 are constants. When $\alpha_2 = \alpha_3 = 0$, X_1 and X_2 are independent. He computed Pitman ARE's for several underlying distributions when U and V are from the same family of univariate distributions.

Table 1.1. Pitman ARE's Computed by Konijn

	Normal	Uniform	Parabolic	Laplace
ARE(ρ, r)	$9/\pi^2$	1	0.8569	1.2656
ARE(τ, r)	$9/\pi^2$	1	0.8569	1.2656
ARE(q', r)	$4/\pi^2$	1/4	0.3164	1

Here the Pitman ARE is defined as the limiting ratio of sample sizes needed for competing tests to maintain the same limiting power. The ARE's obtained under the normal distribution agree with those obtained by Blomqvist (1950) for q' and by Stuart (1954) for ρ and τ. Farlie (1960) also reported the same ARE's as those above when the underlying distribution is bivariate normal.

Next we consider the multivariate test for this problem involving many random vectors (generally more than two) instead of just two univariate random variables. The parametric statistic to consider first for this problem is based on the likelihood ratio criterion derived by Wilks (1935). Suppose the $\boldsymbol{X}_i$'s are multivariate normal with $p \times 1$ mean vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2', \ldots, \boldsymbol{\mu}_c')'$ and $p \times p$ covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \cdots & \boldsymbol{\Sigma}_{1c} \\ \boldsymbol{\Sigma}_{12}' & \boldsymbol{\Sigma}_{22} & \cdots & \boldsymbol{\Sigma}_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{\Sigma}_{1c}' & \boldsymbol{\Sigma}_{2c}' & \cdots & \boldsymbol{\Sigma}_{cc} \end{pmatrix},$$

where $\boldsymbol{\mu}_\alpha$ is $r_\alpha \times 1$ vector and $\boldsymbol{\Sigma}_{\alpha\beta}$ is $r_\alpha \times r_\beta$ covariance matrix. The null hypothesis of independence is equivalent to H_0: $\boldsymbol{\Sigma}_{\alpha\beta} = \boldsymbol{0}_{r_\alpha \times r_\beta}$, $\alpha \neq \beta$. If $\boldsymbol{A} = \sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'$, and we partition it into $\boldsymbol{A}_{\alpha\beta} = \sum_{i=1}^{n}(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}^{(\alpha)})(\boldsymbol{X}_i^{(\beta)} - \bar{\boldsymbol{X}}^{(\beta)})'$, where $\bar{\boldsymbol{X}}^{(\alpha)} = 1/n \sum_{i=1}^{n} \boldsymbol{X}_i^{(\alpha)}$ (similarly defined for $\bar{\boldsymbol{X}}^{(\beta)}$), $\alpha, \beta = 1, 2, \ldots, c$, then the criterion is expressed as

$$V^{n/2} = \left[\frac{|\boldsymbol{A}|}{\prod_{\alpha=1}^{c}|\boldsymbol{A}_{\alpha\alpha}|}\right]^{n/2},$$

(see Anderson 1984, p379). The asymptotic null distribution of $-2\log V^{n/2}$ ($= -n\log V$) is a chi-square distribuion with degrees of freedom of $r_1r_2+r_1r_3+\cdots+r_{c-1}r_c$ (See Anderson,1984). The criterion $-n\log V$ is invariant under the group of transformations

$$\mathcal{G} = \{g(\boldsymbol{D}, \boldsymbol{b}) \mid g(\boldsymbol{D}, \boldsymbol{b})(\boldsymbol{X}) = \boldsymbol{D}\boldsymbol{X} + \boldsymbol{b}\}$$

where $\boldsymbol{D} = \oplus_{\alpha=1}^{c}\boldsymbol{D}_\alpha$, ($\boldsymbol{D}_\alpha$ nonsingular $r_\alpha \times r_\alpha$) and $\boldsymbol{b} \in \mathbb{R}^{\sum_{\alpha=1}^{c} r_\alpha}$, described by Muirhead (1982). Hence the test using $-n\log V$ will not depend on the underlying covariance structures of the $\boldsymbol{X}_i^{(\alpha)}$'s, $\alpha = 1, \ldots, c$.

A nonparametric procedure for this problem was proposed by Puri and Sen (1971). The statistic they proposed is based on component-wise ranking and is defined as

$$S^J = \left[\frac{|\boldsymbol{T}|}{\prod_{\alpha=1}^{c}|\boldsymbol{T}_{\alpha\alpha}|}\right]^{n/2},$$

where

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \cdots & \boldsymbol{T}_{1c} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \cdots & \boldsymbol{T}_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{T}_{c1} & \boldsymbol{T}_{c2} & \cdots & \boldsymbol{T}_{cc} \end{pmatrix}.$$

The elements of $\boldsymbol{T}_{\alpha,\beta} = \{T_{s_\alpha s_\beta}\}_{r_\alpha \times r_\beta}$ are given by

$$T_{s_\alpha s_\beta} = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{R_{s_\alpha i}^{(\alpha)}}{n+1}\right) J\left(\frac{R_{s_\beta i}^{(\beta)}}{n+1}\right),$$

where $R_{s_\alpha i}^{(\alpha)}$ is the rank of $X_{s_\alpha i}^{(\alpha)}$ among $X_{s_\alpha 1}^{(\alpha)}, \ldots, X_{s_\alpha n}^{(\alpha)}$, $R_{s_\beta i}^{(\beta)}$ is the rank of $X_{s_\beta i}^{(\beta)}$ among $X_{s_\beta 1}^{(\beta)}, \ldots, X_{s_\beta n}^{(\beta)}$. Here J is an arbitary nondecreasing and nonconstant score function defined on (0,1) satisfying $\int_0^1 J(u)du = 0$ and $\int_0^1 J^2(u)du = 1$. Under the null hypothesis, $-n\log S^J \xrightarrow{d} \chi^2_{r_1 r_2 + r_1 r_3 + \cdots + r_1 r_c}$, which is the same limiting null distribution as that of $-n\log V$.

We are particularly interested in the score function given by

$$J_0(u) = \begin{pmatrix} 1 & \text{if } 1/2 < \text{u} \leq 1 \\ 0 & \text{if u} = 1/2 \\ -1 & \text{if } 0 \leq \text{u} < 1/2 \end{pmatrix} \tag{1.1}$$

because it produces the test which is an analogue of the sign test. Then using this score function gives the elements of $\boldsymbol{T}_{\alpha,\beta}$ as

$$T_{s_\alpha s_\beta} = \frac{1}{n}\sum_{i=1}^{n} \operatorname{sgn}\left(X_{s_\alpha i}^{(\alpha)} - \widetilde{X}_{s_\alpha}^{(\alpha)}\right) \operatorname{sgn}\left(X_{s_\beta i}^{(\beta)} - \widetilde{X}_{s_\beta}^{(\beta)}\right),$$

where $\widetilde{X}_{s_\alpha}^{(\alpha)}$ $(\widetilde{X}_{s_\beta}^{(\beta)})$ is the median of the $X_{s_\alpha i}^{(\alpha)}$'s $(X_{s_\beta i}^{(\beta)}$'s). However, unlike the statistic $-n \log V$, $-n \log S^{J_0}$ does not have the invariance property because it is based on a component-wise ranking. Thus the perfomance of the test based on $-n \log S^{J_0}$ will depend on the structure of the covariance of the subvectors, the $\boldsymbol{X}_i^{(\alpha)}$'s, $\alpha = 1, \ldots, c$.

Recently, Geiser (1993) proposed the interdirection quadrant statistic extending the bivariate test of independence between two random variables to a multivariate test of independence between two random vectors. Assume that $\hat{\boldsymbol{\theta}}_1$ based on $\boldsymbol{X}_1^{(1)}, \ldots, \boldsymbol{X}_n^{(1)}$ and $\hat{\boldsymbol{\theta}}_2$ based on $\boldsymbol{X}_1^{(2)}, \ldots, \boldsymbol{X}_n^{(2)}$ are affine equivariant (under the group $\mathcal{G}$) estimators of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, respectively, such that both $(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1)$ and $(\hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2)$ are $O_p(n^{-1/2})$. The test statistic he proposed is defined as

$$\hat{Q}_n(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2) = \frac{r_1 r_2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \cos(\pi \hat{p}_1(\boldsymbol{X}_i^{(1)}, \boldsymbol{X}_j^{(1)}; \hat{\boldsymbol{\theta}}_1)) \cos(\pi \hat{p}_2(\boldsymbol{X}_i^{(2)}, \boldsymbol{X}_j^{(2)}; \hat{\boldsymbol{\theta}}_2)),$$

where $\hat{p}_k(\boldsymbol{X}_i^{(k)}, \boldsymbol{X}_j^{(k)}; \hat{\boldsymbol{\theta}}_k)$ is the interdirection proportion between $(\boldsymbol{X}_i^{(k)} - \hat{\boldsymbol{\theta}}_k)$ and $(\boldsymbol{X}_j^{(k)} - \hat{\boldsymbol{\theta}}_k)$, k = 1,2, defined by Randles (1989). Notice that when $r_1 = r_2 = 1$, $\hat{Q}_n(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2)$ reduces to $(\sqrt{n}q')^2$. The model of dependence which Geiser used in exploring the performance of this test procedure is given by

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}^{(1)} \\ \boldsymbol{X}^{(2)} \end{pmatrix} = \begin{pmatrix} (1-\Delta)\boldsymbol{Y}^{(1)} + \Delta \boldsymbol{M}_1 \boldsymbol{Y}^{(2)} \\ \Delta \boldsymbol{M}_2 \boldsymbol{Y}^{(1)} + (1-\Delta)\boldsymbol{Y}^{(2)} \end{pmatrix},$$

where $\boldsymbol{Y}^{(1)}$ and $\boldsymbol{Y}^{(2)}$ are independent random vectors that are $r_1 \times 1$ and $r_2 \times 1$, respectively, and $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are arbitrary (known) matrices of dimensions $r_1 \times r_2$ and $r_2 \times r_1$, respectively. The independence case is produced by setting $\Delta = 0$. When the distributions of both $\boldsymbol{Y}^{(1)}$ and $\boldsymbol{Y}^{(2)}$ are assumed to be from the same elliptically symmetric family. Geiser computed Pitman ARE's among $\hat{Q}_n$, $-n \log V$ and $-n \log S^{J_0}$ by letting $\Delta \to 0$ as $n \to \infty$. When $r_1 = r_2 = 1$, his results agree with those obtained by Konijin (1956). The interdirection quadrant statistic will be

the basis for constructing the statistic for testing independence among many vectors which will be defined in the next chapter.

Now we consider tests for the multivariate multi-sample location problem. We assume that $\boldsymbol{X}_i^{(\alpha)} = (\boldsymbol{X}_{i1}^{(\alpha)}, \ldots, \boldsymbol{X}_{ip}^{(\alpha)})'$, i = 1,..., n_α, denotes a random sample of size n_α from a p-variate continuous population with p-dimensional location parameter $\boldsymbol{\theta}_\alpha$. Here $\alpha = 1, \ldots, c$ indexes samples from c different populations. Theses samples are assumed to be mutually independent. We are interested in testing $H_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_c = \boldsymbol{\theta}$ against the general alternative that $\boldsymbol{\theta}_1$, ... ,$\boldsymbol{\theta}_c$ are not all equal.

The normal theory test for this problem is based on likelihood ratio criterion. Assume that the underlying distributions are all p-variate normal with a common unknown covariance matrix Σ and mean vectors possibly different from each other. The generalized likelihood ratio criterion is given by $U_N = |\boldsymbol{W}|/|\boldsymbol{T}|$, where

$$\boldsymbol{W} = \sum_{\alpha=1}^{c} \sum_{i=1}^{n_\alpha} \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}_\alpha\right) \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}_\alpha\right)'$$

and

$$\boldsymbol{T} = \sum_{\alpha=1}^{c} \sum_{i=1}^{n_\alpha} \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}\right) \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}\right)',$$

with $\bar{\boldsymbol{X}}_\alpha = \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} \boldsymbol{X}_i^{(\alpha)}$, $\bar{\boldsymbol{X}} = \frac{1}{N} \sum_{\alpha=1}^{c} \sum_{i=1}^{n_\alpha} \boldsymbol{X}_i^{(\alpha)}$, and N = $n_1 + n_2 + \cdots + n_c$. It can be shown that $-N \log U_N$ converges, in probability, to Lawley-Hotelling's generalized T^2 (Lawley, 1938 and Hotelling, 1951) defined as

$$T^2 = \sum_{\alpha=1}^{c} n_\alpha \left(\bar{\boldsymbol{X}}_\alpha - \bar{\boldsymbol{X}}\right)' \boldsymbol{S}^{-1} \left(\bar{\boldsymbol{X}}_\alpha - \bar{\boldsymbol{X}}\right),$$

where

$$\boldsymbol{S} = \frac{1}{N-c} \sum_{\alpha=1}^{c} \sum_{i=1}^{n_\alpha} \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}\right) \left(\boldsymbol{X}_i^{(\alpha)} - \bar{\boldsymbol{X}}\right)',$$

The Lawley-Hotelling's generalized T^2 is invariant with respect to the nonsingular linear transformations of the observations $\boldsymbol{X}_i$'s. That is, if $\boldsymbol{D}$ is any nonsingular

$p \times p$ matrix, then the value of T^2 based on the observed vectors $\boldsymbol{DX}_1, \ldots, \boldsymbol{DX}_n$'s is equal to the value of T^2 based on the observed vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. We shall call this invariance property affine-invariance. This invariance property ensures that rotations and reflections of the observations about the origin as well as scale changes will not affect the test statistic. Also the asymptotic distribution of the Lawley-Hotelling's generalized T^2 is chi-square with $p(c-1)$ degrees of freedom. When $c = 2$, Lawley-Hotelling's generalized T^2 reduces to the well-known two-sample Hotelling T^2

$$T^2 = \frac{n_1 n_2}{N} \left(\bar{\boldsymbol{X}}^{(1)} - \bar{\boldsymbol{X}}^{(2)}\right)' \boldsymbol{S}^{-1} \left(\bar{\boldsymbol{X}}^{(1)} - \bar{\boldsymbol{X}}^{(2)}\right),$$

where N $= n_1 + n_2$, $\bar{\boldsymbol{X}}^{(1)} = \frac{1}{n_1} \sum_{i=1}^{n_1} \boldsymbol{X}_i^{(1)}$, $\bar{\boldsymbol{X}}^{(2)} = \frac{1}{n_2} \sum_{j=1}^{n_2} \boldsymbol{X}_j^{(2)}$, and

$$\boldsymbol{S} = \frac{1}{N-2} \left[\sum_{i=1}^{n_1} \left(\boldsymbol{X}_i^{(1)} - \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X}_i^{(1)} - \bar{\boldsymbol{X}}\right)' + \sum_{j=1}^{n_2} \left(\boldsymbol{X}_i^{(2)} - \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X}_i^{(2)} - \bar{\boldsymbol{X}}\right)' \right].$$

Puri and Sen (1971) proposed a nonparametric approach to this problem based on a component-wise ranking. Let $R_{si}^{(\alpha)}$ denote the rank of $X_{si}^{(\alpha)}$ among N observations $X_{s1}^{(1)}, \ldots, X_{sn_1}^{(1)}, \ldots, X_{s1}^{(c)}, \ldots, X_{sn_c}^{(c)}$ for each component $s, s = 1, ..., p$. The $p \times N$ rank matrix is then:

$$\boldsymbol{R} = \begin{pmatrix} R_{11}^{(1)} & \cdots & R_{1n_1}^{(1)} & \cdots & R_{1n_c}^{(c)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{p1}^{(1)} & \cdots & R_{pn_1}^{(1)} & \cdots & R_{pn_c}^{(c)}. \end{pmatrix}.$$

Let $J(\cdot)$ denote a score function defined on (0,1) that is nondecreasing and nonconstant, and satisfies $\int_0^1 J(u)du = 0$ plus $\int_0^1 J^2(u)du < \infty$. Let $E_{R_{si}}^{(\alpha)} = J\left(\frac{R_{si}^{(\alpha)}}{N+1}\right)$ and form the $p \times N$ matrix of scores

$$\boldsymbol{E} = \begin{pmatrix} E_{R_{11}^{(1)}}^{(1)} & \cdots & E_{R_{1n_1}^{(1)}}^{(1)} & \cdots & E_{R_{1n_c}^{(c)}}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{R_{p1}^{(1)}}^{(p)} & \cdots & E_{R_{pn_1}^{(1)}}^{(p)} & \cdots & E_{R_{pn_c}^{(c)}}^{(p)}. \end{pmatrix}.$$

Consider the average rank scores for each s of the c samples, defined by

$$T_s^{(\alpha)} = \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} E^{(s)}_{R^{(\alpha)}_{sn_\alpha}}, \alpha = 1, ..., c; s = 1, .., p.$$

Then the statistic has the form

$$L_N = \sum_{\alpha=1}^{c} n_\alpha \left(\boldsymbol{T}^{(\alpha)} - \bar{\boldsymbol{E}} \right)' \boldsymbol{V}^{-1} \left(\boldsymbol{T}^{(\alpha)} - \bar{\boldsymbol{E}} \right),$$

where $\boldsymbol{T}^{(\alpha)} = (T_1^{(\alpha)}, \ldots, T_p^{(\alpha)})$, and the elements of $\bar{\boldsymbol{E}} = \{\bar{E}^{(s)}\}$ are given by $\bar{E}^{(s)} =$ average of sth row of $\boldsymbol{E}$. Here the elements of $\boldsymbol{V} = \{V_{st}\}$ are given by

$$V_{st} = \frac{1}{N}(\boldsymbol{E}^{(s)\prime} \boldsymbol{E}^{(t)}) - \bar{E}^{(s)} \bar{E}^{(t)},$$

where $\boldsymbol{E}^{(s)}$ ($\boldsymbol{E}^{(t)}$) is the sth (tth) row of $\boldsymbol{E}$. This statistic is analogous to Lawley-Hotelling's generalized T^2 except that scores of ranks are used. Under the null hypothesis, L_N has a limiting chi-square distribution with $p(c-1)$ degrees of freedom. However, unlike T^2, L_N is not affine invariant. We will concentrate attention on the score function in (1.1).

There is another nonparametric procedure based on an affine-invariant test statistic developed quite recently by Hettmansperger and Oja (1994). This statistic is constructed using two-sample location model test statistics and is defined as

$$H = \sum_{k=1}^{c} (1 - n_k/N) H_k.$$

where H_k is the two-sample test for testing the kth sample against all the other samples combined as one. The statistic H_k is obtained from the criterion introduced by Oja (1983) which is

$$T(\boldsymbol{\theta}) = \binom{N}{p}^{-1} \sum_{1 \le i_1 \le \cdots i_p \le N} p! V(\boldsymbol{X}_{i_1}, \ldots, \boldsymbol{X}_{i_p}, \boldsymbol{\theta}),$$

where $V(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_p, \boldsymbol{\theta})$ is the volume of the simplex defined by $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_p$ and $\boldsymbol{\theta}$ and $N = n_1 + n_2$. Here

$$V(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_p, \boldsymbol{\theta}) = \text{abs}\left\{ \frac{1}{p!}\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_1 & X_{11} & \cdots & X_{p1} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_p & X_{1p} & \cdots & X_{pp} \end{pmatrix} \right\}.$$

The Oja multivariate median, $\hat{\boldsymbol{\theta}}$, is the value of $\boldsymbol{\theta}$ that minimizes $T(\boldsymbol{\theta})$. Define the aligned observations by $\boldsymbol{Z}_i = \boldsymbol{X}_i^{(1)} - \hat{\boldsymbol{\theta}}$ for $i = 1, \ldots, n_1$, and $\boldsymbol{Z}_i = \boldsymbol{X}_i^{(2)} - \hat{\boldsymbol{\theta}}$ for $i = n_1 + 1, \ldots, N$. Then the two-sample location model test statistic is defined by $H_k = \boldsymbol{Q}'\boldsymbol{S}^{-1}\boldsymbol{Q}$, where $\boldsymbol{Q} = \sum_{i=1}^{N} a_i \boldsymbol{q}_i$ with $a_i = +1$ for $i = 1, \ldots, n_1$ and $a_i =$ -1 for $i = n_1 + 1, \ldots, N$, and $\boldsymbol{S} = \frac{4n_1n_2}{N(N-1)} \sum_{i=1}^{N} \boldsymbol{q}_i \boldsymbol{q}_i'$. Here

$$\boldsymbol{q}_i = \binom{N-1}{p-1}^{-1} \sum_{1 \le i_1 \le \cdots i_{p-1} \le N} \text{sgn}\left\{ \det \begin{pmatrix} z_{i1} & z_{i_1 1} & \cdots & z_{i_{p-1}1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{ip} & z_{i_1 p} & \cdots & z_{i_{p-1}p} \end{pmatrix} \right\} \begin{pmatrix} \boldsymbol{B}_1(i_1, \ldots, i_{p-1}) \\ \vdots \\ \boldsymbol{B}_p(i_1, \ldots, i_{p-1}) \end{pmatrix}$$

with $\boldsymbol{B}_j(i_1, \ldots, i_{p-1}) =$ - cofactor of z_{ij} from the matrix in the equation above. The asymptotic null distribution of H is chi-square with $p(c-1)$ degrees of freedom. However, the asymptotic relative efficiencies are not yet available to assess the performance of their procedure. However, for the special case $c = 2$, the ARE of H relative to the two-sample Hotelling's T^2 is shown to be the same as that for the multivariate one-sample sign test proposed by Hettmansperger, Nyblom and Oja (1994).

Randles and Peters (1990) proposed a multivariate affine-invariant family of rank tests for the two sample location problem i.e. $c = 2$. This class of statistics is built upon Randles' (1989) multivariate one-sample statistic based on interdirections and the multivariate one-sample signed-rank statistic of Peters and Randles (1990). Let the estimate $\bar{\bar{\boldsymbol{X}}}$ be the sample mean of the N observations. Define the distances to be

$$D_{1,i} = (\boldsymbol{X}_i^{(1)} - \bar{\bar{\boldsymbol{X}}})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{X}_i^{(1)} - \bar{\bar{\boldsymbol{X}}}), \; D_{2,j} = (\boldsymbol{X}_j^{(2)} - \bar{\bar{\boldsymbol{X}}})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{X}_j^{(2)} - \bar{\bar{\boldsymbol{X}}})$$

for i = 1,...,n_1, j = 1,...,n_2 where

$$\hat{\Sigma} = \frac{1}{N}\left\{\sum_{i=1}^{n_1}(\boldsymbol{X}_i^{(1)} - \bar{\bar{\boldsymbol{X}}})(\boldsymbol{X}_i^{(1)} - \bar{\bar{\boldsymbol{X}}})' + \sum_{j=1}^{n_2}(\boldsymbol{X}_j^{(2)} - \bar{\bar{\boldsymbol{X}}})(\boldsymbol{X}_j^{(2)} - \bar{\bar{\boldsymbol{X}}})'\right\}.$$

The two sample test statistic is of the form

$$\begin{aligned} \boldsymbol{Z}_{N,\phi} = \frac{n_1 n_2}{NE(\phi^2)}\Bigg\{ & \frac{1}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\cos\left(\pi\hat{p}_1(i,j;\bar{\bar{\boldsymbol{X}}})\right)\phi\left(\frac{R_{1,i}}{N}\right)\phi\left(\frac{R_{1,j}}{N}\right) \\ & + \frac{1}{n_2^2}\sum_{i=1}^{n_2}\sum_{j=1}^{n_2}\cos\left(\pi\hat{p}_2(i,j;\bar{\bar{\boldsymbol{X}}})\right)\phi\left(\frac{R_{2,i}}{N}\right)\phi\left(\frac{R_{2,j}}{N}\right) \\ & - \frac{2}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\cos\left(\pi\hat{p}_{1,2}(i,j\bar{\bar{\boldsymbol{X}}})\right)\phi\left(\frac{R_{1,i}}{N}\right)\phi\left(\frac{R_{2,j}}{N}\right)\Bigg\} \end{aligned}$$

where $R_{1,i}$ ($R_{2,i}$) denotes the rank of distance $D_{1,i}$ ($D_{2,i}$) among all N distances $D_{1,1}, \ldots, D_{1,n_1}, D_{2,1}, \ldots, D_{2,n_2}$, and ϕ is a nondecreasing score function defined on (0,1) satisfying $0 < E(\phi^2) \equiv \int_0^1 \phi^2(u) < \infty$. Here $\hat{p}_1(i,j;\bar{\bar{\boldsymbol{X}}})$ is the proportion of hyperplanes formed by $p-1$ of $\boldsymbol{X}_{i'}^{(1)}$ and $\bar{\bar{\boldsymbol{X}}}$ such that $\boldsymbol{X}_i^{(1)}$ and $\boldsymbol{X}_j^{(1)}$ are on opposite sides of the hyperplane formed. The term $\hat{p}_2(i,j;\bar{\bar{\boldsymbol{X}}})$ is similary computed from the $\boldsymbol{X}_{i'}^{(2)}$. The term $\hat{p}_{1,2}(i,j;\bar{\bar{\boldsymbol{X}}})$ has a slightly different meaning. The two samples are combined and the hyperplanes are formed by using $p-1$ of the observations from the combined sample and $\bar{\bar{\boldsymbol{X}}}$. The term $\hat{p}_{1,2}(i,j;\bar{\bar{\boldsymbol{X}}})$ is the proportion of these hyperplanes for which $\boldsymbol{X}_i^{(1)}$ and $\boldsymbol{X}_j^{(2)}$ are on opposite sides of the hyperplane formed. The score functions which Randles and Peters used are $\phi_1(u) = u$ and $\phi_2(u) = 1$. The asymptotic null distribution of $\boldsymbol{Z}_{N,\phi}$ is chi-square with p degrees of freedom when the underlying distributions are elliptically symmetric. Also they obtained the asymptotic relative efficiencies relative to the two-sample Hotelling's T^2 under elliptically symmetric underlying distributions for each score function. The asymptotic relative efficiencies turned out to be same as the ones for the one-sample location problem computed by Randles (1989) and Peters and Randles (1990).

A class of test statistics for the multivariate multi-sample location problem will be defined in chapter 5. It will be built by extending the statistic for the multivariate two-sample location problem to the multi-sample setting.

In chapter 2, we define the statistic for testing independence among vectors. The asymptotic distributions of the test under H_0 and under certain contiguous alternatives are also developed in chapter 2. The Pitman asymptotic relative efficiencies are presented in chapter 3. In chapter 4 we make comparisons among several competing procedures via Monte Carlo studies and apply the proposed test to some real data. A class of tests for the multivariate multi-sample location problem are defined in chapter 5. The asymptotic distributions of the tests under H_0 and under certain contiguous alternatives are also developed and the asymptotic relative efficiencies of the proposed tests relative to Lawley-Hotelling's generalized T^2 are evaluated. Finally, in chapter 6, we show Monte Carlo results and an example to see the effectiveness of the proposed tests.

CHAPTER 2
A MULTIVARIATE TEST FOR INDEPENDENCE AMONG VECTORS

2.1 Definition of the Test Statistic

The nonparametric sign statistic based on interdirections, called the interdirection quadrant statistic and denoted by $\hat{Q}_n$, was proposed by Gieser (1993) for testing whether the two vectors are independent. In this disssertation we will extend the case of two vectors to the case of many vectors. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be independent and identically distributed vectors with $\boldsymbol{X}_i = (\boldsymbol{X}_i^{(1)\prime}, \boldsymbol{X}_i^{(2)\prime}, \ldots, \boldsymbol{X}_i^{(c)\prime})'$. Here $\boldsymbol{X}_i^{(\alpha)}$ is a $r_\alpha \times 1$ vector, so $\boldsymbol{X}_i$ is a $(\sum_{\alpha=1}^{c} r_\alpha) \times 1$ vector. We assume $\boldsymbol{X}$ has a continuous distribution with density function $f_X\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(c)}\right)$. The vector $\boldsymbol{X}_i^{(\alpha)}$ has a marginal density $f_\alpha\left(\boldsymbol{x}^{(\alpha)}\right)$, $\alpha = 1, 2, \ldots, c$. Here $f_\alpha\left(\boldsymbol{x}^{(\alpha)}\right)$ represents the density of an elliptically symmetric distribution centered at the $r_\alpha \times 1$ vector $\boldsymbol{\theta}_\alpha$. We would like to test

$$H_0\colon f_X\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(c)}\right) = f_1\left(\boldsymbol{x}^{(1)}\right) f_2\left(\boldsymbol{x}^{(2)}\right) \cdots f_c\left(\boldsymbol{x}^{(c)}\right)$$

versus

$$H_1\colon f_X\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(c)}\right) \neq f_1\left(\boldsymbol{x}^{(1)}\right) f_2\left(\boldsymbol{x}^{(2)}\right) \cdots f_c\left(\boldsymbol{x}^{(c)}\right).$$

The test statistic we propose is defined as $\hat{Q}_n^{1,2} + \hat{Q}_n^{1,3} + \cdots + \hat{Q}_n^{c-1,c}$, where $\hat{Q}_n^{\alpha,\beta}$ is the interdirection quadrant statistic computed between $\boldsymbol{X}^{(\alpha)}$ and $\boldsymbol{X}^{(\beta)}$, for each $1 \leq \alpha < \beta \leq c$. That is, the proposed statistic is the sum of $\frac{(c-1)c}{2}$ interdirection quadrant statistics. Let $\hat{\boldsymbol{\theta}}_\alpha$ be an affine equivariant estimator of $\boldsymbol{\theta}_\alpha$ based on $\boldsymbol{X}_1^{(\alpha)}, \ldots, \boldsymbol{X}_n^{(\alpha)}$ that satisfies $(\hat{\boldsymbol{\theta}}_\alpha - \boldsymbol{\theta}_\alpha)$ is $O_p(n^{-1/2})$. Then our extention of the interdirection quadrant

statistic, which we shall denote $\hat{Q}_n^T$, is given by

$$\hat{Q}_n^T(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \ldots, \hat{\boldsymbol{\theta}}_c) =$$

$$\sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \left[\frac{r_\alpha r_\beta}{n} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cos(\pi \hat{p}_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}; \hat{\boldsymbol{\theta}}_\alpha)) \cos(\pi \hat{p}_\beta(\boldsymbol{X}_{i_1}^{(\beta)}, \boldsymbol{X}_{i_2}^{(\beta)}; \hat{\boldsymbol{\theta}}_\beta)) \right].$$

where

$$\hat{p}_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}; \hat{\boldsymbol{\theta}}_\alpha) = \begin{pmatrix} (C_{i_1,i_2}^{(\alpha)} + d_n)/\binom{n}{p-1} & \text{if } i_1 \neq i_2 \\ 0 & \text{if } i_1 = i_2, \end{pmatrix}$$

$$d_n = \frac{1}{2}\left[\binom{n}{p-1} - \binom{n-2}{p-1}\right]$$

and the interdirection count $C_{i_1,i_2}^{(\alpha)}$, first defined by Randles (1989), denotes the number of hyperplanes defined by the origin and r_α-1 observations $\boldsymbol{X}_i^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$ (excluding $\boldsymbol{X}_{i_1}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$ and $\boldsymbol{X}_{i_2}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$) such that $\boldsymbol{X}_{i_1}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$, and $\boldsymbol{X}_{i_2}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$ are on opposite sides of the hyperplane. The interdirection counts are used to measure the angular distance between the centered vectors $\boldsymbol{X}_{i_1}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$ and $\boldsymbol{X}_{i_2}^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}_\alpha$ relative to the origin and the positions of the other observations. The term $\hat{p}_\beta(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}; \hat{\boldsymbol{\theta}}_\beta)$ is similarly defined among the $\boldsymbol{X}_i^{(\beta)}$ - $\hat{\boldsymbol{\theta}}_\beta$ vectors. We note that $\hat{Q}_n^T$ is invariant since the interdirection count is invariant under the group $\mathcal{G}$. In the next section we will derive the asymptotic null distribution of $\hat{Q}_n^T$ in the cases where $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c$ are assumed known and unknown, respectively.

2.2 Asymptotic Null Distribution of $\hat{Q}_n^T$

We shall define the class of elliptically symmetric distributions before deriving the asymptotic null distribution of $\hat{Q}_n^T$.

Definition 2.2.1 The $r_\alpha \times 1$ random vector $\boldsymbol{X}^{(\alpha)}$ is said to have an elliptically symmetric distribution with parameters $\boldsymbol{\theta}_\alpha(r_\alpha \times 1)$ and $\boldsymbol{\Sigma}_\alpha(r_\alpha \times r_\alpha)$ if the density function

of $\boldsymbol{X}^{(\alpha)}$ is of the form

$$f_\alpha(\boldsymbol{X}^{(\alpha)}) = K_\alpha|\boldsymbol{\Sigma}_\alpha|^{-1/2}g_\alpha[(\boldsymbol{X}^{(\alpha)} - \boldsymbol{\theta}_\alpha)'\boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{X}^{(\alpha)} - \boldsymbol{\theta}_\alpha)], \tag{2.1}$$

where g_α is a non-negative, real-valued, differentiable function, $\boldsymbol{\Sigma}_\alpha$ is positive definite and symmetric and K_α is a positive scalar such that (2.1) represents a density function.

An elliptically symmetric distribution with dispersion parameter $\sigma^2\boldsymbol{I}_\alpha$ is called a spherically symmetric distribution. Since $\hat{Q}_n^T$ is affine invariant and an elliptically symmetric distribution is transformable into a spherically symmetric distribution through a nonsingular linear transformation, we assume without loss of generality that $f_\alpha(\boldsymbol{X}^{(\alpha)})$ is a spherically symmetric distribution centered at the origin.

In order to find the asymptotic distribution of $\hat{Q}_n^T$ under H_0, we construct an approximating quantity which has the same asymptotic distribution. We first consider the case where $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c$ are known. Under the assumed underlying distribution, we define

$$Q_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c) = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\left[\frac{r_\alpha r_\beta}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cos(\pi p_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}))\cos(\pi p_\beta(\boldsymbol{X}_{i_1}^{(\beta)}, \boldsymbol{X}_{i_2}^{(\beta)}))\right]$$

This $Q_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c)$ is the same as the statistic $\hat{Q}_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c)$ except that the estimated proportion is replaced with $p_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}) = \mathrm{E}_{H_0}\left[\hat{p}_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}) \mid \boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}\right]$ = (radian measure of the angle between $\boldsymbol{X}_{i_1}^{(\alpha)}$ and $\boldsymbol{X}_{i_2}^{(\alpha)}$) / π. Now let $\boldsymbol{X}_i^{(\alpha)} = R_i^{(\alpha)}\boldsymbol{U}_i^{(\alpha)}$ where $R_i^{(\alpha)} = [\boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{X}_i^{(\alpha)}]^{1/2}$. Note that $\boldsymbol{U}^{(\alpha)}$ is distributed uniformly on the r_α dimensional unit hypersphere and is independent of the positive quantity $R^{(\alpha)}$. The estimated proportion of hyperplanes, $\hat{p}_\alpha$, and p_α only require the directions of $\boldsymbol{U}_{i_1}^{(\alpha)}$ and $\boldsymbol{U}_{i_2}^{(\alpha)}$, not their lengths so that $\cos(\pi p_\alpha(\boldsymbol{X}_{i_1}^{(\alpha)}, \boldsymbol{X}_{i_2}^{(\alpha)}))$ = cos(angle between $\boldsymbol{U}_{i_1}^{(\alpha)}$ and $\boldsymbol{U}_{i_2}^{(\alpha)}$) $= \boldsymbol{U}_{i_1}^{(\alpha)\prime}\boldsymbol{U}_{i_2}^{(\alpha)}$. Thus we have

$$Q_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c) = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \left[\frac{r_\alpha r_\beta}{n} \sum_{i_\alpha=1}^{n} \sum_{i_\beta=1}^{n} \boldsymbol{U}_{i_\alpha}^{(\alpha)\prime} \boldsymbol{U}_{i_\beta}^{(\alpha)} \boldsymbol{U}_{i_\alpha}^{(\beta)\prime} \boldsymbol{U}_{i_\beta}^{(\beta)} \right]$$

$$= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \left[\frac{1}{n} \sum_{s_\alpha=1}^{r_\alpha} \sum_{s_\beta=1}^{r_\beta} \left(\sum_{j=1}^{n} \sqrt{r_\alpha r_\beta} U_{s_\alpha j}^{(\alpha)} U_{s_\beta j}^{(\beta)} \right)^2 \right], \tag{2.2}$$

where $U_{s_\alpha j}^{(\alpha)}$ ($U_{s_\beta j}^{(\beta)}$) is the s_αth (s_βth) component of $U_j^{(\alpha)}$ ($U_j^{(\beta)}$). We are now prepared to state the theorem which determines the asymptotic null distribution of $Q_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c)$.

<u>Theorem 2.2.1</u> Under H_0, $Q_n^T(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_c) \xrightarrow{d} \chi^2_{r_1 r_2 + r_1 r_3 + \cdots + r_{c-1} r_c}$.

<u>Proof of Theorem 2.2.1</u> Let $\boldsymbol{B} = \boldsymbol{B}_{1,2} \oplus \boldsymbol{B}_{1,3} \oplus \cdots \oplus \boldsymbol{B}_{c-1,c}$ where $\boldsymbol{B}_{1,2} = \{b_{s_1 s_2}\}_{r_1 \times r_2}$, $\boldsymbol{B}_{1,3} = \{b_{s_1 s_3}\}_{r_1 \times r_3}$, $\cdots$, $\boldsymbol{B}_{c-1,c} = \{b_{s_{c-1} s_c}\}_{r_{c-1} \times r_c}$ are arbitrary nonzero but fixed matrices. Define $\boldsymbol{Z} = \boldsymbol{Z}_{1,2} \oplus \boldsymbol{Z}_{1,3} \oplus \cdots \oplus \boldsymbol{Z}_{c-1,c}$ where $\boldsymbol{Z}_{1,2} = \{\sum_{j=1}^n \sqrt{r_1 r_2} U_{s_1 j}^{(1)} U_{s_2 j}^{(2)}\}_{r_1 \times r_2}$, $\boldsymbol{Z}_{1,3} = \{\sum_{j=1}^n \sqrt{r_1 r_3} U_{s_1 j}^{(1)} U_{s_3 j}^{(3)}\}_{r_1 \times r_3}, \cdots, \boldsymbol{Z}_{c-1,c} = \{\sum_{j=1}^n \sqrt{r_{c-1} r_c} U_{s_{c-1} j}^{(c-1)} U_{s_c j}^{(c)}\}_{r_{c-1} \times r_c}$. Then

$$\begin{aligned}
\operatorname{vec}(\boldsymbol{B})' \operatorname{vec}(\boldsymbol{Z}) &= \operatorname{vec}(\boldsymbol{B}_{1,2})' \operatorname{vec}(\boldsymbol{Z}_{1,2}) + \operatorname{vec}(\boldsymbol{B}_{1,3})' \operatorname{vec}(\boldsymbol{Z}_{1,3}) + \cdots \\
&\quad + \operatorname{vec}(\boldsymbol{B}_{c-1,c})' \operatorname{vec}(\boldsymbol{Z}_{c-1,c}) \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{s_\alpha=1}^{r_\alpha} \sum_{s_\beta=1}^{r_\beta} b_{s_\alpha s_\beta} Z_{s_\alpha s_\beta} \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{j=1}^{n} \left[\sum_{s_\alpha=1}^{r_\alpha} \sum_{s_\beta=1}^{r_\beta} \sqrt{r_\alpha r_\beta} b_{s_\alpha s_\beta} U_{s_\alpha j}^{(\alpha)} U_{s_\beta j}^{(\beta)} \right] \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{j=1}^{n} \boldsymbol{U}_j^{(\alpha)\prime} (\sqrt{r_\alpha r_\beta} \boldsymbol{B}_{\alpha,\beta}) \boldsymbol{U}_j^{(\beta)} \\
&= \sum_{j=1}^{n} \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \boldsymbol{U}_j^{(\alpha)\prime} (\sqrt{r_\alpha r_\beta} \boldsymbol{B}_{\alpha,\beta}) \boldsymbol{U}_j^{(\beta)}
\end{aligned}$$

The summands in the last formula are the sum of iid random variables with mean zero and variance $\operatorname{vec}(\boldsymbol{B}_{\alpha,\beta})' \operatorname{vec}(\boldsymbol{B}_{\alpha,\beta})$, and the covariance between any two terms are

zero. Thus by the usual central limit theorem $n^{-1/2}\text{vec}\,(\boldsymbol{B})'\,\text{vec}\,(\boldsymbol{Z}) \sim AN(0, \text{vec}\,(\boldsymbol{B})'\,\text{vec}\,(\boldsymbol{B}))$. It follows that $n^{-1}\text{vec}\,(\boldsymbol{Z})'\,\text{vec}\,(\boldsymbol{Z}) \xrightarrow{d} \chi^2_{r_1r_2+r_1r_3+\cdots+r_{c-1}r_c}$. Since $n^{-1}\text{vec}\,(\boldsymbol{Z})'\,\text{vec}\,(\boldsymbol{Z})$ $= n^{-1}\Big(\sum_{s_1=1}^{r_1}\sum_{s_2=1}^{r_2} Z^2_{1,2}(s_1,s_2) + \sum_{s_1=1}^{r_1}\sum_{s_3=1}^{r_3} Z^2_{1,3}(s_1,s_3) + \cdots + \sum_{s_{c-1}=1}^{r_{c-1}}\sum_{s_c=1}^{r_c}$ $Z^2_{c-1,c}(s_{c-1},s_c)\Big) = Q_n^T(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_c)$ in (2.2), where $Z_{\alpha,\beta}(s_\alpha,s_\beta)$ is the element in the s_αth row and s_βth column of $\boldsymbol{Z}_{\alpha,\beta}$, the result is proved.

Now using the result $\hat{Q}_n^{\alpha,\beta}(\boldsymbol{\theta}_\alpha,\boldsymbol{\theta}_\beta) = Q_n^{\alpha,\beta}(\boldsymbol{\theta}_\alpha,\boldsymbol{\theta}_\beta) + o_p(1)$, in Geiser (1993), we are able to find the asymptotic null distribution of $\hat{Q}_n^T(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_c)$.

<u>Theorem 2.2.2</u> Under H_0, $\hat{Q}_n^T(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_c) \xrightarrow{d} \chi^2_{r_1r_2+r_1r_3+\cdots+r_{c-1}r_c}$.

<u>Proof of Theorem 2.2.2</u>

$$\begin{aligned}
\hat{Q}_n^T - Q_n^T &= \hat{Q}_n^{1,2} + \hat{Q}_n^{1,3} + \cdots + \hat{Q}_n^{c-1,c} - Q_n^{1,2} - Q_n^{1,3} - \cdots - Q_n^{c-1,c} \\
&= (\hat{Q}_n^{1,2} - Q_n^{1,2}) + (\hat{Q}_n^{1,3} - Q_n^{1,3}) + \cdots + (\hat{Q}_n^{c-1,c} - Q_n^{c-1,c}) \\
&= o_p(1).
\end{aligned}$$

Thus Theorem 2.2.1 gives the desired result.

Next we consider the case in which $\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\cdots\boldsymbol{\theta}_c$ are unknown and hence estimated. The asymptotic null distribution is easily obtained using the result, $\hat{Q}_n^{\alpha,\beta}(\hat{\boldsymbol{\theta}}_\alpha,\hat{\boldsymbol{\theta}}_\beta) = \hat{Q}_n^{\alpha,\beta}(\boldsymbol{\theta}_\alpha,\boldsymbol{\theta}_\beta) + o_p(1)$, by Geiser and following the same procedure as in Theorem 2.2.2 above. Thus the asymptotic null distribution of $\hat{Q}_n^T$ remains the same whether the symmetric center is known or not, provided the center is estimated using a $\hat{\boldsymbol{\theta}}_\alpha$ with the stated properties. Since $\hat{Q}_n^T$ has the same asymptotic null distribution as $-n\log V$ and $-n\log S^{J_0}$, we can observe the relative perfomance among them by comparing their efficiencies. Pitman relative efficiency will be used to make these comparisons.

2.3 Asymptotic Distribution of $\hat{Q}_n^T$ under Contiguous Alternatives

As the first step in obtaining Pitman asymptotic relative efficiencies, we will find the asymptotic distribution of $\hat{Q}_n^T$ under a sequence of alternatives approaching the null hypothesis. In doing so, we first will propose a model that shows a dependence among the components of the observations. In this model we express the dependence as a function of a non-negative real-valued parameter Δ. The sequence of alternatives defined by the model converges to the null hypothesis as $\Delta \to 0$ in such a way that the alternatives are contiguous to the null hypothesis. We will use the model obtained by generalizing the one which Konijin (1956) studied. The general multivariate version of this model is given by

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}^{(1)} \\ \boldsymbol{X}^{(2)} \\ \vdots \\ \boldsymbol{X}^{(c)} \end{pmatrix} = \begin{pmatrix} (1-(c-1)\Delta)\boldsymbol{Y}^{(1)} + \Delta\boldsymbol{M}_{1,2}\boldsymbol{Y}^{(2)} + \cdots + \Delta\boldsymbol{M}_{1,c}\boldsymbol{Y}^{(c)} \\ \Delta\boldsymbol{M}_{2,1}\boldsymbol{Y}^{(1)} + (1-(c-1)\Delta)\boldsymbol{Y}^{(2)} + \cdots + \Delta\boldsymbol{M}_{2,c}\boldsymbol{Y}^{(c)} \\ \vdots \\ \Delta\boldsymbol{M}_{c,1}\boldsymbol{Y}^{(1)} + \Delta\boldsymbol{M}_{c,2}\boldsymbol{Y}^{(2)} + \cdots + (1-(c-1)\Delta)\boldsymbol{Y}^{(c)} \end{pmatrix}$$

$$= \begin{pmatrix} (1-(c-1)\Delta)\boldsymbol{I}_{r_1} & \Delta\boldsymbol{M}_{1,2} & \cdots & \Delta\boldsymbol{M}_{1,c} \\ \Delta\boldsymbol{M}_{2,1} & (1-(c-1)\Delta)\boldsymbol{I}_{r_2} & \cdots & \Delta\boldsymbol{M}_{2,c} \\ \vdots & \vdots & & \vdots \\ \Delta\boldsymbol{M}_{c,1} & \Delta\boldsymbol{M}_{c,2} & \cdots & (1-(c-1)\Delta)\boldsymbol{I}_{r_c} \end{pmatrix} \begin{pmatrix} \boldsymbol{Y}^{(1)} \\ \boldsymbol{Y}^{(2)} \\ \vdots \\ \boldsymbol{Y}^{(c)} \end{pmatrix}$$

$$= \boldsymbol{A}_\Delta \begin{pmatrix} \boldsymbol{Y}^{(1)} \\ \boldsymbol{Y}^{(2)} \\ \vdots \\ \boldsymbol{Y}^{(c)} \end{pmatrix} = \boldsymbol{A}_\Delta \boldsymbol{Y}, \tag{2.3}$$

where $\boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(2)}, \ldots, \boldsymbol{Y}^{(c)}$ are independent random vectors with dimensions of r_1,r_2, $\ldots$,r_c, respectively and $\boldsymbol{M}_{1,2}, \boldsymbol{M}_{1,3}, \ldots, \boldsymbol{M}_{c,c-1}$ are fixed non-zero matrices of dimensions $r_1 \times r_2$, $r_1 \times r_3$, $\ldots$, $r_c \times r_{c-1}$, respectively, and $0 \leq \Delta \leq 1/c$. Assume $\boldsymbol{A}_\Delta$ is nonsingular for Δ in a neighborhood of 0. Note that $\Delta = 0$ corresponds to the null hypothesis of independence. We assume that each $\boldsymbol{Y}^{(\alpha)}$ is elliptically symmetric with zero mean vector and variance covariance matrix $\boldsymbol{\Sigma}_\alpha$, $\alpha = 1, \ldots, c$. With these underlying distributions, we will apply LeCam's three lemmas as descrived in Hajek and Sidak (1967, pp. 201-214) to find the asymptotic distribution. The sequence of alternatives $H_1 : \Delta_n = n^{-1/2}\Delta_0$, where $\Delta_0 > 0$, can be shown to be contiguous to the null hypothesis by following the same arguments as in Geiser (1993). This contiguity helps us find the asymptotic distribution by approximating the log-likelihood function. Finding the asymptotic distribution under the contiguous alternatives involves establishing the asymptotic bivariate normality of the appropriately defined statistic and the log-likelihood function Λ_n under H_0 where $\Lambda_n = \sum_{i=1}^n \log L(\boldsymbol{X}_i; \Delta_n)$, and $L(\boldsymbol{x}; \Delta_n) = f_X(\boldsymbol{x}; \Delta_n)/f_X(\boldsymbol{x}; 0)$. Under H_0 the log-likelihood function is approximated by $T_n = \Delta_n \sum_{i=1}^n L'(\boldsymbol{X}_i; 0)$ where $L'(\boldsymbol{x}; 0) \equiv \frac{\partial}{\partial \Delta} L(\boldsymbol{x}; \Delta)|_{\Delta=0}$. LeCam's third lemma states that if, under H_0, $\binom{S_n}{\Lambda_n} \sim AN\left(\binom{\mu_1}{-\sigma_2^2/2}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$, then $S_n \sim AN(\mu_1 + \sigma_{12}, \sigma_1^2)$ under a contiguous sequence of alternatives. Thus, obtaining the asymptotic bivariate normality of (S_n, T_n) under H_0 yields the asymptotic normality of S_n under the contiguous alternatives. First we shall find the expression for $L'(\boldsymbol{X}_i; 0)$ and then establish the asymptotic normallity of T_n.

Lemma 2.3.1 For Model (1.1)

$$
\begin{aligned}
L'(\boldsymbol{x};0) &= 2(c-1)\left(\boldsymbol{x}^{(1)\prime}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{x}^{(1)}\phi_1(\boldsymbol{x}^{(1)\prime}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{x}^{(1)}) + \frac{r_1}{2}\right)\\
&\quad + 2(c-1)\left(\boldsymbol{x}^{(2)\prime}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{x}^{(2)}\phi_2(\boldsymbol{x}^{(2)\prime}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{x}^{(2)}) + \frac{r_2}{2}\right)\\
&\quad + \cdots\\
&\quad + 2(c-1)\left(\boldsymbol{x}^{(c)\prime}\boldsymbol{x}^{(c)}\phi_c(\boldsymbol{x}^{(c)\prime}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{x}^{(c)}) + \frac{r_c}{2}\right)\\
&\quad - 2\boldsymbol{x}^{(1)\prime}\left(\phi_1(\boldsymbol{x}^{(1)\prime}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{x}^{(1)})\boldsymbol{\Sigma}_1^{-1}\boldsymbol{M}_{1,2} + \phi_2(\boldsymbol{x}^{(2)\prime}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{x}^{(2)})\boldsymbol{M}_{2,1}'\boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{x}^{(2)}\\
&\quad - 2\boldsymbol{x}^{(1)\prime}\left(\phi_1(\boldsymbol{x}^{(1)\prime}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{x}^{(1)})\boldsymbol{\Sigma}_1^{-1}\boldsymbol{M}_{1,3} + \phi_3(\boldsymbol{x}^{(3)\prime}\boldsymbol{\Sigma}_3^{-1}\boldsymbol{x}^{(3)})\boldsymbol{M}_{3,1}'\boldsymbol{\Sigma}_3^{-1}\right)\boldsymbol{x}^{(3)}\\
&\quad - \cdots\\
&\quad - 2\boldsymbol{x}^{(c-1)\prime}\left(\phi_{c-1}(\boldsymbol{x}^{(c-1)\prime}\boldsymbol{\Sigma}_{c-1}^{-1}\boldsymbol{x}^{(c-1)})\boldsymbol{\Sigma}_{c-1}^{-1}\boldsymbol{M}_{c-1,c} + \phi_c(\boldsymbol{x}^{(c)\prime}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{x}^{(c)})\boldsymbol{M}_{c,c-1}'\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{x}^{(c)},
\end{aligned}
$$

where $\phi_\alpha(t) = g_\alpha'(t)/g_\alpha(t)$.

Proof of Lemma 2.3.1 Since the density function of $\boldsymbol{X}$ is

$$f_X(\boldsymbol{x};\Delta) = \operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right) f_Y\left(\boldsymbol{A}_\Delta^{-1}\boldsymbol{x}\right),$$

where $f_Y(\boldsymbol{y}) = f_1\left(\boldsymbol{y}^{(1)}\right) f_2\left(\boldsymbol{y}^{(2)}\right)\cdots f_c\left(\boldsymbol{y}^{(c)}\right)$, we have

$$L(\boldsymbol{x};\Delta) = \operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right)\frac{f_Y\left(\boldsymbol{A}_\Delta^{-1}\boldsymbol{x}\right)}{f_Y(\boldsymbol{x})},$$

and hence

$$L'(\boldsymbol{x};\Delta) = \left(\frac{\partial}{\partial\Delta}\operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right)\right)\frac{f_Y\left(\boldsymbol{A}_\Delta^{-1}\boldsymbol{x}\right)}{f_Y(\boldsymbol{x})} + \operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right)\left(\frac{\partial}{\partial\Delta}\frac{f_Y\left(\boldsymbol{A}_\Delta^{-1}\boldsymbol{x}\right)}{f_Y(\boldsymbol{x})}\right).$$

Using

$$\frac{\partial}{\partial\Delta}\operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right) = -\operatorname{abs}\left(|\boldsymbol{A}_\Delta|^{-1}\right)\operatorname{tr}\left(-\boldsymbol{A}_\Delta^{-1}\boldsymbol{P}\right),$$

where

$$P \equiv \frac{\partial}{\partial \Delta} A_\Delta = \begin{pmatrix} (c-1)I_{r_1} & -M_{1,2} & \cdots & -M_{1,c} \\ -M_{2,1} & (c-1)I_{r_2} & \cdots & -M_{2,c} \\ & & \vdots & \\ -M_{c,1} & -M_{c,2} & \cdots & (c-1)I_{r_c} \end{pmatrix},$$

we have

$$\begin{aligned} \frac{\partial}{\partial \Delta} \text{abs}\left(|A_\Delta|^{-1}\right)\bigg|_{\Delta=0} &= \text{abs}\left(|A_0|^{-1}\right) \text{tr}\left(A_0^{-1} P\right) \\ &= \text{abs}\left(|I_{r_1+r_2+\cdots+r_c}|^{-1}\right) \text{tr}\left(I_{r_1+r_2+\cdots+r_c}^{-1} P\right) \\ &= \text{tr}(P) = (r_1 + r_2 + \cdots + r_c)(c-1). \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial}{\partial \Delta} \frac{f_Y\left(A_\Delta^{-1} x\right)}{f_Y(x)} &= \left(\frac{\partial}{\partial \Delta} A_\Delta^{-1} x\right)' \frac{\nabla f_Y(x)}{f_Y(x)} \\ &= \left(A_\Delta^{-1} P A_\Delta^{-1} x\right)' \frac{\nabla f_Y(x)}{f_Y(x)}, \end{aligned}$$

so that

$$\left.\frac{f_Y\left(\boldsymbol{A}_\Delta^{-1}\boldsymbol{x}\right)}{f_Y\left(\boldsymbol{x}\right)}\right|_{\Delta=0} = \left(\boldsymbol{A}_0^{-1}\boldsymbol{P}\boldsymbol{A}_0^{-1}\boldsymbol{x}\right)'\frac{\nabla f_Y\left(\boldsymbol{x}\right)}{f_Y\left(\boldsymbol{x}\right)}$$

$$= \left(\boldsymbol{I}_{r_1+r_2+\cdots+r_c}^{-1}\boldsymbol{P}\boldsymbol{I}_{r_1+r_2+\cdots+r_c}^{-1}\boldsymbol{x}\right)'\frac{\nabla f_Y\left(\boldsymbol{x}\right)}{f_Y\left(\boldsymbol{x}\right)}$$

$$= (\boldsymbol{P}\boldsymbol{x})'\frac{\nabla f_Y\left(\boldsymbol{x}\right)}{f_Y\left(\boldsymbol{x}\right)}$$

$$= \left(\begin{pmatrix} (c-1)\boldsymbol{I}_{r_1} & -\boldsymbol{M}_{1,2} & \cdots & -\boldsymbol{M}_{1,c} \\ -\boldsymbol{M}_{2,1} & (c-1)\boldsymbol{I}_{r_2} & \cdots & -\boldsymbol{M}_{2,c} \\ & & \vdots & \\ -\boldsymbol{M}_{c,1} & -\boldsymbol{M}_{c,2} & \cdots & (c-1)\boldsymbol{I}_{r_c} \end{pmatrix}\begin{pmatrix}\boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \\ \vdots \\ \boldsymbol{x}^{(c)}\end{pmatrix}\right)'$$

$$\times \begin{pmatrix} \dfrac{\nabla f_1\left(\boldsymbol{x}^{(1)}\right)}{f_1\left(\boldsymbol{x}^{(1)}\right)} \\ \dfrac{\nabla f_2\left(\boldsymbol{x}^{(2)}\right)}{f_2\left(\boldsymbol{x}^{(2)}\right)} \\ \vdots \\ \dfrac{\nabla f_2\left(\boldsymbol{x}^{(c)}\right)}{f_c\left(\boldsymbol{x}^{(c)}\right)} \end{pmatrix}$$

$$= 2((c-1)\boldsymbol{x}^{(1)} - \boldsymbol{M}_{1,2}\boldsymbol{x}^{(2)} - \cdots - \boldsymbol{M}_{1,c}\boldsymbol{x}^{(c)})'\Sigma_1^{-1}\boldsymbol{x}^{(1)}\phi_1(\boldsymbol{x}^{(1)\prime}\Sigma_1^{-1}\boldsymbol{x}^{(1)})$$

$$+ 2(-\boldsymbol{M}_{2,1}\boldsymbol{x}^{(1)} + (c-1)\boldsymbol{x}^{(2)} - \cdots - \boldsymbol{M}_{2,c}\boldsymbol{x}^{(c)})'\Sigma_2^{-1}\boldsymbol{x}^{(2)}\phi_2(\boldsymbol{x}^{(2)\prime}\Sigma_2^{-1}\boldsymbol{x}^{(2)})$$

$$+ \cdots$$

$$+ 2(-\boldsymbol{M}_{c,1}\boldsymbol{x}^{(1)} - \boldsymbol{M}_{c,2}\boldsymbol{x}^{(2)} - \cdots + (c-1)\boldsymbol{x}^{(c)})'$$

$$\Sigma_c^{-1}\boldsymbol{x}^{(c)}\phi_c(\boldsymbol{x}^{(c)\prime}\Sigma_c^{-1}\boldsymbol{x}^{(c)}),$$

because

$$\frac{\nabla f_\alpha\left(\boldsymbol{x}^{(\alpha)}\right)}{f_\alpha\left(\boldsymbol{x}^{(\alpha)}\right)} = 2\boldsymbol{x}^{(\alpha)}\,\frac{g_\alpha'\left(\boldsymbol{x}^{(\alpha)\prime}\boldsymbol{x}^{(\alpha)}\right)}{g_\alpha\left(\boldsymbol{x}^{(\alpha)\prime}\boldsymbol{x}^{(\alpha)}\right)}.$$

This completes the Lemma.

Next we use the relation $\boldsymbol{X}^{(\alpha)} = R^{(\alpha)}\boldsymbol{U}^{(\alpha)}$ and make a transformation by multiplying by a nonsingular matrix $\boldsymbol{D}_\alpha$, where $\Sigma_\alpha = \boldsymbol{D}_\alpha \boldsymbol{D}'_\alpha$ in order to get the representation of the elliptically symmetric distribution from the spherically symmetric distribution. Then using the form $R^{(\alpha)}\boldsymbol{D}_\alpha \boldsymbol{U}^{(\alpha)}$ for $\boldsymbol{X}^{(\alpha)}$ gives

$$
\begin{aligned}
T_n = n^{-1/2}\sum_{i=1}^{n} 2\Delta_0 \Big\{ & (c-1)\Big((R_i^{(1)})^2\phi_1((R_i^{(1)})^2) + \frac{r_1}{2}\Big) \\
& + (c-1)\Big((R_i^{(2)})^2\phi_2((R_i^{(2)})^2) + \frac{r_2}{2}\Big) \\
& + \cdots \\
& + (c-1)\Big((R_i^{(c)})^2\phi_c((R_i^{(c)})^2) + \frac{r_c}{2}\Big) \\
& - \boldsymbol{U}_i^{(1)\prime}\boldsymbol{H}_i^{1,2}\boldsymbol{U}_i^{(2)} - \boldsymbol{U}_i^{(1)\prime}\boldsymbol{H}_i^{1,3}\boldsymbol{U}_i^{(3)} - \cdots - \boldsymbol{U}_i^{(c-1)\prime}\boldsymbol{H}_i^{c-1,c}\boldsymbol{U}_i^{(c)} \Big\},
\end{aligned}
$$

where

$$
\boldsymbol{H}_i^{\alpha,\beta} = R_i^{(\alpha)} R_i^{(\beta)} \boldsymbol{D}_\alpha{}' \left(\phi_\alpha((R_i^{(\alpha)})^2)\Sigma_\alpha^{-1}\boldsymbol{M}_{\alpha,\beta} + \phi_\beta((R_i^{(\beta)})^2)\boldsymbol{M}'_{\beta,\alpha}\Sigma_\beta^{-1}\right)\boldsymbol{D}_\beta.
$$

This expression of T_n is a generalization of the case based on two vectors in Geiser (1993), i.e. $T_n = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} T_n^{\alpha,\beta}$, where

$$
\begin{aligned}
T_n^{\alpha,\beta} = n^{-1/2}\sum_{i=1}^{n} 2\Delta_0 \Big\{ & \Big((R_i^{(\alpha)})^2\phi_\alpha((R_i^{(\alpha)})^2) + \frac{r_\alpha}{2}\Big) \\
& + \Big((R_i^{(\beta)})^2\phi_\beta((R_i^{(\beta)})^2) + \frac{r_\beta}{2}\Big) - \boldsymbol{U}_i^{(\alpha)\prime}\boldsymbol{H}_i^{\alpha,\beta}\boldsymbol{U}_i^{(\beta)} \Big\}.
\end{aligned}
$$

He showed that $T_n^{\alpha,\beta} \sim AN(0,\sigma^{*2})$, where $\mathrm{V}_{H_0}\left[T_n^{\alpha,\beta}\right] \equiv \sigma^{*2} < \infty$ under the assumptions that $\mathrm{E}_{H_0}\left[(R^{(k)})^4\phi_k^2((R^{(k)})^2)\right] < \infty$, $\mathrm{E}_{H_0}\left[(R^{(k)})^2\phi_k^2((R^{(k)})^2)\right] < \infty$, and $\mathrm{E}_{H_0}\left[(R^{(k)})^2\right] < \infty$. Using the same arguments, we also have $T_n \sim AN(0,\sigma^2)$, where $\mathrm{V}_{H_0}[T_n] \equiv \sigma^2 < \infty$, by the usual central limit theorem. Now we are in the position of deriving the asymptotic distribution of $\hat{Q}_n^T$ under Δ_n. Since $\hat{Q}_n^T$ is affine invariant, we put $\Sigma_\alpha = \boldsymbol{I}_\alpha$ without loss of generality.

<u>Theorem 2.3.1</u> Under Δ_n,

$$\hat{Q}_n^T \xrightarrow{d} \chi^2_{r_1 r_2 + r_1 r_3 + \cdots + r_{c-1} r_c}(\lambda_1),$$

where

$$\lambda_1 = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \frac{4\Delta_0^2}{r_\alpha r_\beta} \operatorname{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)' \operatorname{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)$$

<u>Proof of Theorem 2.3.1</u> Let $\boldsymbol{a} = (a_1, a_2)'$ be an arbitrary vector of constants none of which are zero. Let $S_n = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} S_n^{\alpha,\beta}$, where

$$S_n^{\alpha,\beta} = n^{-1/2} \sum_{i=1}^{n} \boldsymbol{U}_i^{(\alpha)'} (\sqrt{r_\alpha r_\beta} \boldsymbol{B}_{\alpha,\beta}) \boldsymbol{U}_i^{(\beta)}$$

Then

$$\boldsymbol{a}' \begin{pmatrix} S_n \\ T_n \end{pmatrix} =$$

$$\sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \Bigg[n^{-1/2} \sum_{i=1}^{n} 2\Delta_0 \Bigg\{ a_2 \left((R_i^{(\alpha)})^2 \phi_1((R_i^{(\alpha)})^2) + \frac{r_\alpha}{2} \right) + a_2 \left((R_i^{(\beta)})^2 \phi_2((R_i^{(\beta)})^2) + \frac{r_\beta}{2} \right)$$

$$+ \boldsymbol{U}_i^{(\alpha)'} \left(a_2 \boldsymbol{H}_i^{\alpha,\beta} + a_1 \frac{\sqrt{r_\alpha r_\beta}}{2\Delta_0} \boldsymbol{B}_{\alpha,\beta} \right) \boldsymbol{U}_i^{(\beta)} \Bigg\} \Bigg].$$

Since $\mathrm{E}_{H_0}\left[S_n^{\alpha,\beta}\right] = 0$ and $\mathrm{E}_{H_0}\left[T_n^{\alpha,\beta}\right] = 0$, it follows that $\mathrm{E}_{H_0}\left[\boldsymbol{a}'(S_n, T_n)'\right] = 0$. Also

$$
\begin{aligned}
V_{H_o}\{\boldsymbol{a}'(S_n, T_n)\} &= V_{H_o}\{a_1(S_n^{1,2} + S_n^{1,3} + \cdots + S_n^{c-1,c}) + a_2 T_n\} \\
&= a_1^2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} V_{H_o}(S_n^{\alpha,\beta}) + a_2^2 V_{H_o}(T_n) \\
&\quad + 2a_1 a_2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \mathrm{Cov}_{H_0}\left[S_n^{\alpha,\beta}, T_n\right] \\
&= a_1^2 \Big\{ \mathrm{vec}\,(\boldsymbol{B}_{1,2})' \,\mathrm{vec}\,(\boldsymbol{B}_{1,2}) + \mathrm{vec}\,(\boldsymbol{B}_{1,3})' \,\mathrm{vec}\,(\boldsymbol{B}_{1,3}) + \cdots \\
&\quad + \mathrm{vec}\,(\boldsymbol{B}_{c-1,c})' \,\mathrm{vec}\,(\boldsymbol{B}_{c-1,c}) \Big\} \\
&\quad + a_2^2 \sigma^2 \\
&\quad + 2a_1 a_2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} 2\Delta_0 \Big\{ \sqrt{r_\alpha r_\beta}\, \mathrm{E}_{H_0}\left[\boldsymbol{U}^{(\alpha)'} \boldsymbol{H}^{\alpha,\beta} \boldsymbol{U}^{(\beta)} \boldsymbol{U}^{(\alpha)'} \boldsymbol{B}_{\alpha,\beta} \boldsymbol{U}^{(\beta)}\right] \Big\},
\end{aligned}
$$

where $\sigma^2 = V(T_n)$. The second equality follows from the fact that $\mathrm{Cov}_{H_0}\left[S_n^{\alpha,\beta}, T_n^{\alpha',\beta'}\right] = 0$ when one or more of the superscripts is unique. Also since

$$
\begin{aligned}
\mathrm{E}_{H_0}\left[\boldsymbol{U}^{(\alpha)'} \boldsymbol{H}^{\alpha,\beta} \boldsymbol{U}^{(\beta)} \boldsymbol{U}^{(\alpha)'} \boldsymbol{B}_{\alpha,\beta} \boldsymbol{U}^{(\beta)}\right] &= \frac{1}{r_\beta} \mathrm{E}_{H_0}\left[\boldsymbol{U}^{(\alpha)'} \boldsymbol{H}^{\alpha,\beta} \boldsymbol{B}_{\alpha,\beta}{}' \boldsymbol{U}^{(\alpha)}\right] \\
&= \frac{1}{r_\alpha r_\beta} \mathrm{vec}\,(\boldsymbol{B}_{\alpha,\beta})' \,\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right),
\end{aligned}
$$

we have $\sigma_1^2 = \mathrm{vec}\,(\boldsymbol{B})' \,\mathrm{vec}\,(\boldsymbol{B})$, $\sigma_2^2 = \sigma^2$, and

$$\sigma_{12} = \begin{pmatrix} \mathrm{vec}\,(\boldsymbol{B}_{1,2}) \\ \mathrm{vec}\,(\boldsymbol{B}_{1,3}) \\ \vdots \\ \mathrm{vec}\,(\boldsymbol{B}_{c-1,c}) \end{pmatrix}' \begin{pmatrix} \frac{2\Delta_0}{\sqrt{r_1 r_2}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,2}\right]\right) \\ \frac{2\Delta_0}{\sqrt{r_1 r_3}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,3}\right]\right) \\ \vdots \\ \frac{2\Delta_0}{\sqrt{r_{c-1} r_c}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{c-1,c}\right]\right) \end{pmatrix}$$

$$= \mathrm{vec}\,(\boldsymbol{B})' \begin{pmatrix} \frac{2\Delta_0}{\sqrt{r_1 r_2}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,2}\right]\right) \\ \frac{2\Delta_0}{\sqrt{r_1 r_3}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,3}\right]\right) \\ \vdots \\ \frac{2\Delta_0}{\sqrt{r_{c-1} r_c}} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{c-1,c}\right]\right) \end{pmatrix}$$

Then the asymptotic normality of $\boldsymbol{a}'(S_n, T_n)'$ follows by applying the usual central limit theorem, and hence, under H_0 $\left(\begin{smallmatrix} S_n \\ T_n \end{smallmatrix}\right) \sim AN\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{smallmatrix}\right)\right)$. Thus under Δ_n,

$$S_n \sim AN\left(\sigma_{12}, \mathrm{vec}\,(\boldsymbol{B})' \,\mathrm{vec}\,(\boldsymbol{B})\right),$$

and

$$Q_n^T \xrightarrow{d} \chi^2_{r_1 r_2 + r_1 r_2 + \cdots + r_{c-1} r_c}(\lambda_1).$$

We note also that Q_n^T and $\hat{Q}_n^T$ have the same asymptotic distribution under Δ_n as well as under H_0 because $|Q_n^T - \hat{Q}_n^T| = o_p(1)$ under Δ_n. Therefore, the result follows.

Next we find the asymptotic distribution of $-n \log V$ and $-n \log S^{J_0}$ under Δ_n. We approximate $-n \log V$ and $-n \log S^{J_0}$ as the sum of statistics, respectively, each of which is based on pairs of vectors. Puri and Sen (1971, pp.352) show that

$$-n \log S^{J_0} = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} -n \log (S^{J_0})^{\alpha,\beta} + O_p(n^{-1})$$

via Laplace's expansion of determinants. Similarly,

$$-n \log V = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} -n \log V^{\alpha,\beta} + O_p(n^{-1})$$

following the same procedure. Then using the expression of each $-n\log V^{\alpha,\beta}$ in Geiser (1993), we have

$$-n\log V = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(R_i^{(\alpha)}R_j^{(\alpha)}R_i^{(\beta)}R_j^{(\beta)}\right)\boldsymbol{U}_i^{(\alpha)\prime}\boldsymbol{U}_j^{(\alpha)}\boldsymbol{U}_i^{(\beta)\prime}\boldsymbol{U}_j^{(\beta)} + o_p(1),$$

Here we also put $\Sigma_\alpha = \boldsymbol{I}_\alpha$ because of the invariance of $-n\log V$. Now we establish the following theorem.

<u>Theorem 2.3.2</u> Under Δ_n,

$$-n\log V \xrightarrow{d} \chi^2_{r_1r_2+r_1r_3+\cdots+r_{c-1}r_c}(\lambda_2)$$

where

$$\lambda_2 = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{4\Delta_0^2}{r_\alpha^2 r_\beta^2}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(\alpha)}R^{(\beta)}\boldsymbol{H}^{\alpha,\beta}\right]\right)'\,\mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(\alpha)}R^{(\beta)}\boldsymbol{H}^{\alpha,\beta}\right]\right)$$

<u>Proof of Theorem 2.3.2</u> Let

$$S_n = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} S_n^{\alpha,\beta},$$

where

$$\begin{aligned} S_n^{\alpha,\beta} &= n^{-1/2}\mathrm{vec}\,(\boldsymbol{B}_{\alpha,\beta})'\,\mathrm{vec}\left(\sum_{i=1}^{n}\boldsymbol{X}_i^{(\alpha)}\boldsymbol{X}_i^{(\beta)\prime}\right) \\ &= n^{-1/2}\sum_{i=1}^{n}\boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{B}_{\alpha,\beta}\boldsymbol{X}_i^{(\beta)} \\ &= n^{-1/2}\sum_{i=1}^{n}\boldsymbol{U}_i^{(\alpha)\prime}(R_i^{(\alpha)}R_i^{(\beta)}\boldsymbol{B}_{\alpha,\beta})\boldsymbol{U}_i^{(\beta)} \end{aligned}$$

Since

$$\begin{aligned}
\sigma_{12} &= \mathrm{Cov}_{H_0}\left[S_n, T_n\right] \\
&= \mathrm{Cov}_{H_0}\left[S_n^{1,2} + S_n^{1,3} + \cdots + S_n^{c-1,c}, T_n\right] \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \mathrm{Cov}_{H_0}\left[S_n^{\alpha,\beta}, T_n^{\alpha,\beta}\right] \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} 2\Delta_0 \mathrm{E}_{H_0}\left[\boldsymbol{U}^{(\alpha)\prime} \boldsymbol{H}^{\alpha,\beta} \boldsymbol{U}^{(\beta)} \boldsymbol{U}^{(\alpha)\prime} R^{(\alpha)} R^{(\beta)} \boldsymbol{B}_{\alpha,\beta} \boldsymbol{U}^{(\beta)}\right] \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \mathrm{vec}\left(\boldsymbol{B}_{\alpha,\beta}\right)' \left(\frac{2\Delta_0}{r_\alpha r_\beta} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(\alpha)} R^{(\beta)} \boldsymbol{H}^{\alpha,\beta}\right]\right)\right) \\
&= \mathrm{vec}\left(\boldsymbol{B}\right)' \begin{pmatrix} \frac{2\Delta_0}{r_1 r_2} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(1)} R^{(2)} \boldsymbol{H}^{1,2}\right]\right) \\ \frac{2\Delta_0}{r_1 r_3} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(1)} R^{(3)} \boldsymbol{H}^{1,3}\right]\right) \\ \vdots \\ \frac{2\Delta_0}{r_{c-1} r_c} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(c-1)} R^{(c)} \boldsymbol{H}^{c-1,c}\right]\right) \end{pmatrix}
\end{aligned}$$

This yields the result.

Similarly following the same reasoning in Geiser and using the result in Puri and Sen (1971,p.359) gives

$$\begin{aligned}
&- n \log S^{J_0} \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} n^{-1} \mathrm{vec}\left(\sum_{i=1}^{n} \mathrm{sgn}\left(\boldsymbol{U}_i^{(\alpha)}\right) \mathrm{sgn}\left(\boldsymbol{U}_i^{(\beta)}\right)'\right)' \mathrm{vec}\left(\sum_{i=1}^{n} \mathrm{sgn}\left(\boldsymbol{U}_i^{(\alpha)}\right) \mathrm{sgn}\left(\boldsymbol{U}_i^{(\beta)}\right)'\right) \\
&\quad + o_p(1) \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{sgn}\left(\boldsymbol{U}_i^{(\alpha)}\right)' \mathrm{sgn}\left(\boldsymbol{U}_j^{(\alpha)}\right) \mathrm{sgn}\left(\boldsymbol{U}_i^{(\beta)}\right)' \mathrm{sgn}\left(\boldsymbol{U}_j^{(\beta)}\right) \\
&\quad + o_p(1),
\end{aligned}$$

where $\mathrm{sgn}(\boldsymbol{U}^{(\alpha)}) = (\mathrm{sgn}(U_1),\ldots,\mathrm{sgn}(U_{r_\alpha}))'$ and J_o is the score function defined by (1.1). For simplicity of calculation we choose to use $\boldsymbol{I}_\alpha$ for Σ_α, although this causes

some loss of generality due to the fact that $-n \log S^{J_0}$ is not affine-invariant. Note however that the use of a spherically symmetric distribution is actually favorable to a component-wise test (see Randles 1989). Next we find the asymptotic distribution of $-n \log S^{J_0}$ under the contiguous alternatives. A well-known result is needed to derive this distribution. If $\boldsymbol{U}^{(\alpha)} \sim \text{Uniform}(\Omega_{r_\alpha})$, then

$$\mathrm{E}\left[|U_t^{(\alpha)}|\right] = \frac{\Gamma\left(\frac{r_\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{r_\alpha+1}{2}\right)}.$$

<u>Theorem 2.3.3</u> Under Δ_n,

$$-n \log S^{J_0} \xrightarrow{d} \chi^2_{r_1 r_2 + r_1 r_3 + \cdots + r_{c-1} r_c}(\lambda_3)$$

where

$$\lambda_3 = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \frac{4\Delta_0^2 \Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2 \Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)} \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)' \mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)$$

<u>Proof of Theorem 2.3.3</u> Let

$$S_n = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} n^{-1/2} \sum_{i=1}^{n} \mathrm{sgn}\left(\boldsymbol{U}_i^{(\alpha)}\right)' \boldsymbol{B}_{\alpha,\beta} \mathrm{sgn}\left(\boldsymbol{U}_i^{(\beta)}\right)$$

Again

$$\begin{aligned}
\sigma_{12} &= \mathrm{Cov}_{H_0}\left[S_n, T_n\right] \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} \mathrm{Cov}_{H_0}\left[S_n^{\alpha,\beta}, T_n^{\alpha,\beta}\right] \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} 2\Delta_0 \mathrm{E}_{H_0}\left[\boldsymbol{U}^{(\alpha)\prime}\boldsymbol{H}^{\alpha,\beta}\boldsymbol{U}^{(\beta)}\mathrm{sgn}\left(\boldsymbol{U}^{(\alpha)}\right)'\boldsymbol{B}_{\alpha,\beta}\mathrm{sgn}\left(\boldsymbol{U}^{(\beta)}\right)\right] \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} \mathrm{vec}\left(\boldsymbol{B}_{\alpha,\beta}\right)'\left(2\Delta_0 \mathrm{E}_{H_0}\left[|U_1^{(\alpha)}|\right]\mathrm{E}_{H_0}\left[|U_1^{(\beta)}|\right]\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)\right) \\
&= \mathrm{vec}\left(\boldsymbol{B}\right)'\begin{pmatrix}
\dfrac{2\Delta_0\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}{\pi\Gamma\left(\frac{r_1+1}{2}\right)\Gamma\left(\frac{r_2+1}{2}\right)}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,2}\right]\right) \\
\dfrac{2\Delta_0\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_3}{2}\right)}{\pi\Gamma\left(\frac{r_1+1}{2}\right)\Gamma\left(\frac{r_3+1}{2}\right)}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{1,3}\right]\right) \\
\vdots \\
\dfrac{2\Delta_0\Gamma\left(\frac{r_{c-1}}{2}\right)\Gamma\left(\frac{r_c}{2}\right)}{\pi\Gamma\left(\frac{r_{c-1}+1}{2}\right)\Gamma\left(\frac{r_c+1}{2}\right)}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{c-1,c}\right]\right)
\end{pmatrix}
\end{aligned}$$

The result follows.

Based on the distributions of three statistics in Theorems 2.3.1, 2.3.2. and 2.3.3, we will find the relative perfomances between them by computing their ARE's for certain specified distributions.

CHAPTER 3
COMPARISONS OF STATISTICS

3.1 Pitman Asymptotic Relative Efficiencies

In this section we will use Pitman asymptotic relative efficiencies (ARE) to make comparisons between $\hat{Q}_n^T$ and $-n\log V$ and between $\hat{Q}_n^T$ and $-n\log S^{J_0}$. Since each of the statistics has asymptotic noncentral chisquare distribution with degrees of freedom of $r_1r_2 + r_1r_3 + \cdots + r_{c-1}r_c$ under Δ_n, the Pitman ARE is the ratio of the noncentral parameters (see Hannan, 1956). We will apply the asymptotic results under the contiguous alternatives in the previous section. Using Theorem 2.3.1 and Theorem 2.3.2 we find

$$\begin{aligned}
&\mathrm{ARE}(\hat{Q}_n^T, -n\log V)\\
&= \frac{\lambda_1}{\lambda_2}\\
&= \frac{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{4\Delta_0^2}{r_\alpha r_\beta}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)'\mathrm{vec}\left(\mathrm{E}_{H_0}\left[\boldsymbol{H}^{\alpha,\beta}\right]\right)}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{4\Delta_0^2}{r_\alpha^2 r_\beta^2}\mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(\alpha)}R^{(\beta)}\boldsymbol{H}^{\alpha,\beta}\right]\right)'\mathrm{vec}\left(\mathrm{E}_{H_0}\left[R^{(\alpha)}R^{(\beta)}\boldsymbol{H}^{\alpha,\beta}\right]\right)}\\
&= \frac{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{1}{r_\alpha r_\beta}\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)'\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{1}{4}\mathrm{vec}\left(\boldsymbol{M}_{\alpha,\beta}+\boldsymbol{M}'_{\beta,\alpha}\right)'\mathrm{vec}\left(\boldsymbol{M}_{\alpha,\beta}+\boldsymbol{M}'_{\beta,\alpha}\right)}
\end{aligned}$$

where

$$\varphi_{\alpha,\beta} = \mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)}\phi_\alpha((R^{(\alpha)})^2)\right],$$

for $1 \le \alpha < \beta \le c$. For simplicity, we assume that all the elements of $\boldsymbol{M}_{\alpha,\beta}$ and its corresponding matrix $\boldsymbol{M}_{\beta,\alpha}$ are the same constant $d_{\alpha,\beta}$ i.e. $\boldsymbol{M}_{1,2} = \boldsymbol{M}'_{2,1} =$

$\left((d_{1,2})\right)_{r_1\times r_2}$, $\boldsymbol{M}_{1,3}=\boldsymbol{M}'_{3,1}=\left((d_{1,3})\right)_{r_1\times r_3}, \cdots, \boldsymbol{M}_{c-1,c}=\boldsymbol{M}'_{c,c-1}=\left((d_{c-1,c})\right)_{r_{c-1}\times r_c}$.

Then

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log V)=\frac{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2(\varphi_{\alpha,\beta}+\varphi_{\beta,\alpha})^2}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 r_\alpha r_\beta}$$

When $c=2$, $\mathrm{ARE}(\hat{Q}_n^T, -n\log V)$ reduces to $\frac{(\varphi_{1,2}+\varphi_{1,3})^2}{r_1 r_2}$ which is derived by Geiser (1993) under the same assumption. With the further assumption that $d_{1,2}=d_{1,3}=\cdots=d_{c-1,c}$,

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log V)=\frac{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}(\varphi_{\alpha,\beta}+\varphi_{\beta,\alpha})^2}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} r_\alpha r_\beta}.$$

Applying Theorem 2.3.1 and Theorem 2.3.3 we see that

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})=\frac{\lambda_1}{\lambda_3}=\frac{\lambda_1^I}{\lambda_3^I},$$

where

$$\lambda_1^I=\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\frac{1}{r_\alpha r_\beta}\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)'\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)$$

and

$$\lambda_3^I=\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\left[\frac{r_\alpha r_\beta\Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2\Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)}\right.$$

$$\left.\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)'\mathrm{vec}\left(\varphi_{\alpha,\beta}\boldsymbol{M}_{\alpha,\beta}+\varphi_{\beta,\alpha}\boldsymbol{M}'_{\beta,\alpha}\right)\right]$$

Further simplication via setting $\boldsymbol{M}_{\alpha,\beta}=\boldsymbol{M}_{\beta,\alpha}=\left((d_{\alpha,\beta})\right)_{r_\alpha\times r_\beta}$ gives

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})=\frac{\lambda_1^{II}}{\lambda_3^{II}},$$

where

$$\lambda_1^{II}=\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2(\varphi_{\alpha,\beta}+\varphi_{\beta,\alpha})^2$$

and

$$\lambda_3^{II} = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 (\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha})^2 \frac{r_\alpha r_\beta \Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2\Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)}$$

Again the special case in which $c = 2$ gives the same result for $\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})$ as derived in Geiser (1993), i.e.

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0}) = \frac{\pi^2\Gamma^2\left(\frac{r_1+1}{2}\right)\Gamma^2\left(\frac{r_2+1}{2}\right)}{r_1 r_2 \Gamma^2\left(\frac{r_1}{2}\right)\Gamma^2\left(\frac{r_2}{2}\right)}.$$

In general, when $c > 2$, $\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})$ depends on g_α, $\alpha = 1,\ldots c$, and the matrices $\boldsymbol{M}_{\alpha,\beta}$, $1 \leq \alpha < \beta \leq c$ because the $d_{\alpha,\beta}$'s and $\varphi_{\alpha,\beta}$'s in the numerator and denominator do not cancel each other. However, when $r_1 = r_2 = \cdots = r_c \equiv r$, we have

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0}; r) = \frac{\pi^2\Gamma^4\left(\frac{r+1}{2}\right)}{r^2\Gamma^4\left(\frac{r}{2}\right)}, \tag{3.1}$$

for any number of vectors $c \geq 2$. This expression is again of the same form as in the case of two vectors and shows no dependence on the matrices $\boldsymbol{M}_{\alpha,\beta}$ and g_α at all. In other words $\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})$ is a function of r only, independent of the distributions as long as they belong to the elliptically symmetric distributions family. Of course, this is true only when the Σ_α's are all diagonal. In the following sections we will compute $\mathrm{ARE}(\hat{Q}_n^T, -n\log V)$ and $\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0})$ for various choices of $g_1, g_2, \cdots, g_c$.

3.2 Exponential Power Class

We consider an exponential power family of elliptically symmetric distributions with

$$g_\alpha(w) = \exp\left(-\frac{w^{\nu_\alpha}}{c_\alpha}\right),$$

$$K_\alpha = \frac{\nu_\alpha \Gamma\left(\frac{r_\alpha}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)} \left[\frac{\Gamma\left(\frac{r_\alpha+2}{2\nu_\alpha}\right)}{\pi r_\alpha \Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)}\right]^{r_\alpha/2},$$

$$c_\alpha = \left[\frac{r_\alpha \Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)}{\Gamma\left(\frac{r_\alpha+2}{2\nu_\alpha}\right)}\right]^{\nu_\alpha} \equiv t_\alpha^{\nu_\alpha},$$

as given in (2.1) With this density $\boldsymbol{X}^{(\alpha)}$ has the exponential power distribution centered at the zero vector with $\Sigma_{r_\alpha} = \boldsymbol{I}_{r_\alpha}$ and shape parameter ν_α. The notation for this family will be ($\boldsymbol{X}^{(\alpha)} \sim \text{Exp}(\nu_\alpha)$). Using this distibution, we evaluate $\text{ARE}(\hat{Q}_n^T, -n\log V)$ and $\text{ARE}(\hat{Q}_n^T, -n\log S^{J_0})$ with different values of the parameters ν_α. When $\nu_\alpha = 1$, the distribution is a multivariate normal. When $0 < \nu_\alpha < 1$, the resulting distribution has heavier tails than the normal and when $\nu_\alpha > 1$, the distribution has lighter tails than the normal. In fact, as $\nu_\alpha \to \infty$, the distribution approaches that of a uniform distribution inside a ν_α dimensional sphere. In order to evaluate ARE's we need the following useful results

Lemma 3.2.1

$$\mathrm{E}_{H_0}\left[(R^{(\alpha)})^m\right] = \frac{\Gamma\left(\frac{r_\alpha+m}{2\nu_\alpha}\right)}{\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)} t_\alpha^{m/2} \qquad \text{provided } r_\alpha + m > 0,$$

Proof of Lemma 3.2.1 The density function of S = $(R^{(\alpha)})^2$ is given by

$$q(w) = \frac{K_\alpha \pi^{\nu_\alpha/2}}{\Gamma\left(\frac{\nu_\alpha}{2}\right)} w^{r_\alpha/2-1} \exp\left(-\frac{w^{\nu_\alpha}}{c_\alpha}\right). \tag{3.2}$$

Then

$$\mathrm{E}\left[S^{m/2}\right] = \frac{K_\alpha \pi^{\nu_\alpha/2}}{\Gamma\left(\frac{\nu_\alpha}{2}\right)} \int_0^\infty s^{(r_\alpha+m)/2-1} \exp(-(s/t_\alpha)^{\nu_\alpha}) ds$$

$$= \frac{K_\alpha \pi^{\nu_\alpha/2} t_\alpha^{(r_\alpha+m)/2}}{\Gamma\left(\frac{\nu_\alpha}{2}\right)\nu_\alpha} \int_0^\infty u^{(r_\alpha+m)/(2\nu_\alpha)-1} \exp(-u) du$$

$$= \frac{K_\alpha \pi^{\nu_\alpha/2} t_\alpha^{(r_\alpha+m)/2} \Gamma\left(\frac{r_\alpha+m}{2\nu_\alpha}\right)}{\Gamma\left(\frac{\nu_\alpha}{2}\right)\nu_\alpha}$$

$$= \frac{\Gamma\left(\frac{r_\alpha+m}{2\nu_\alpha}\right)}{\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)} t_\alpha^{m/2},$$

so long as $r_\alpha + m > 0$.

Lemma 3.2.2

$$\varphi_{\alpha,\beta} = -\nu_\alpha \left(\frac{t_\beta}{t_\alpha}\right)^{1/2} \frac{\Gamma\left(\frac{r_\beta+1}{2\nu_\beta}\right)\Gamma\left(\frac{r_\alpha+2\nu_\alpha-1}{2\nu_\alpha}\right)}{\Gamma\left(\frac{r_\beta}{2\nu_\beta}\right)\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)},$$

, for $1 \le \alpha < \beta \le c$

Proof of Lemma 3.2.2 Since

$$\phi_\alpha(w) = g'_\alpha(w)/g_\alpha(w) = -\frac{\nu_\alpha}{c_\alpha} w^{\nu_\alpha - 1},$$

$$\begin{aligned}
\varphi_{\alpha,\beta} &= \mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)}\phi_\alpha((R^{(\alpha)})^2)\right] \\
&= -\frac{\nu_\alpha}{c_\alpha}\mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)}R^{(\alpha)^{2(\nu_\alpha-1)}}\right] \\
&= -\frac{\nu_\alpha}{c_\alpha}\mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)^{2\nu_\alpha-1}}\right] \\
&= -\frac{\nu_\alpha}{c_\alpha}\frac{\Gamma\left(\dfrac{r_\beta+1}{2\nu_\beta}\right)}{\Gamma\left(\dfrac{r_\beta}{2\nu_\beta}\right)}t_\beta^{1/2}\cdot\frac{\Gamma\left(\dfrac{r_\alpha+2\nu_\alpha-1}{2\nu_\alpha}\right)}{\Gamma\left(\dfrac{r_\alpha}{2\nu_\alpha}\right)}t_\alpha^{(2\nu_\alpha-1)/2} \\
&= -\nu_\alpha\left(\frac{t_\beta}{t_\alpha}\right)^{1/2}\frac{\Gamma\left(\dfrac{r_\beta+1}{2\nu_\beta}\right)\Gamma\left(\dfrac{r_\alpha+2\nu_\alpha-1}{2\nu_\alpha}\right)}{\Gamma\left(\dfrac{r_\beta}{2\nu_\beta}\right)\Gamma\left(\dfrac{r_\alpha}{2\nu_\alpha}\right)}.
\end{aligned}$$

Using Lemmas 3.2.1 and 3.2.2, we are able to state the following Theorems.

Theorem 3.2.1 If $\boldsymbol{X}^{(\alpha)} \sim \mathrm{Exp}(\nu_\alpha)$, with $\nu_\alpha > -(r_\alpha - 2)/4$, and we assume that $\boldsymbol{M}_{1,2} = \boldsymbol{M}'_{2,1} = \left((d_{1,2})\right)_{r_1\times r_2}$, $\boldsymbol{M}_{1,3} = \boldsymbol{M}'_{3,1} = \left((d_{1,3})\right)_{r_1\times r_3}$, $\cdots$, $\boldsymbol{M}_{c-1,c} = \boldsymbol{M}'_{c,c-1} = \left((d_{c-1,c})\right)_{r_{c-1}\times r_c}$, then

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log V; \nu_1, \nu_2, \cdots, \nu_c) = \frac{\lambda_1^{III}}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 r_\alpha r_\beta},$$

where

$$\begin{aligned}
\lambda_1^{III} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \Bigg(&\nu_\alpha\left(\frac{t_\beta}{t_\alpha}\right)^{1/2}\frac{\Gamma\left(\dfrac{r_\beta+1}{2\nu_\beta}\right)\Gamma\left(\dfrac{r_\alpha+2\nu_\alpha-1}{2\nu_\alpha}\right)}{\Gamma\left(\dfrac{r_\beta}{2\nu_\beta}\right)\Gamma\left(\dfrac{r_\alpha}{2\nu_\alpha}\right)} \\
&+ \nu_\beta\left(\frac{t_\alpha}{t_\beta}\right)^{1/2}\frac{\Gamma\left(\dfrac{r_\alpha+1}{2\nu_\alpha}\right)\Gamma\left(\dfrac{r_\beta+2\nu_\beta-1}{2\nu_\beta}\right)}{\Gamma\left(\dfrac{r_\alpha}{2\nu_\alpha}\right)\Gamma\left(\dfrac{r_\beta}{2\nu_\beta}\right)}\Bigg)^2.
\end{aligned}$$

Corollary 3.2.1 If $\nu_1 = \nu_2 = \cdots = \nu_c \equiv \nu$ and $r_1 = r_2 = \cdots = r_c \equiv r$ then

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log V; \nu; r) = \left[\frac{2\nu\Gamma\left(\frac{r+1}{2\nu}\right)\Gamma\left(\frac{r+2\nu-1}{2\nu}\right)}{r\Gamma^2\left(\frac{r}{2\nu}\right)}\right]^2, \tag{3.3}$$

regardless of values of $d_{\alpha,\beta}$.

Theorem 3.2.2 If $\boldsymbol{X}^{(\alpha)} \sim \mathrm{Exp}(\nu_\alpha)$, with $\nu_\alpha > -(r_\alpha - 2)/4$, and we assume that $\boldsymbol{M}_{1,2} = \boldsymbol{M}'_{2,1} = \left((d_{1,2})\right)_{r_1\times r_2}$, $\boldsymbol{M}_{1,3} = \boldsymbol{M}'_{3,1} = \left((d_{1,3})\right)_{r_1\times r_3}$, $\cdots$, $\boldsymbol{M}_{c-1,c} = \boldsymbol{M}'_{c,c-1} = \left((d_{c-1,c})\right)_{r_{c-1}\times r_c}$, then

$$\mathrm{ARE}(\hat{Q}_n^T, -n\log S^{J_0}) = \frac{\lambda_1^{III}}{\lambda_3^{III}},$$

with

$$\lambda_3^{III} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \left(\nu_\alpha\left(\frac{t_\beta}{t_\alpha}\right)^{1/2}\frac{\Gamma\left(\frac{r_\beta+1}{2\nu_\beta}\right)\Gamma\left(\frac{r_\alpha+2\nu_\alpha-1}{2\nu_\alpha}\right)}{\Gamma\left(\frac{r_\beta}{2\nu_\beta}\right)\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)}\right.$$

$$\left. + \nu_\beta\left(\frac{t_\alpha}{t_\beta}\right)^{1/2}\frac{\Gamma\left(\frac{r_\alpha+1}{2\nu_\alpha}\right)\Gamma\left(\frac{r_\beta+2\nu_\beta-1}{2\nu_\beta}\right)}{\Gamma\left(\frac{r_\alpha}{2\nu_\alpha}\right)\Gamma\left(\frac{r_\beta}{2\nu_\beta}\right)}\right)^2$$

$$\times\left(\frac{r_\alpha r_\beta\Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2\Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)}\right)$$

The expression (3.3) is exactly the same as the one based on two vectors. Thus all the comparisons made by Geiser (1993) for $c = 2$ are valid for any number of vectors

$c \geq 2$. For instances,

$$\text{ARE}(\hat{Q}_n^T, -n\log V; \nu; 1) = \left[\frac{2\nu\Gamma\left(\dfrac{1}{\nu}\right)}{\Gamma^2\left(\dfrac{1}{2\nu}\right)}\right]^2,$$

$$\text{ARE}(\hat{Q}_n^T, -n\log V; 1; r) = \left[\frac{2\Gamma^2\left(\dfrac{r+1}{2}\right)}{r\Gamma^2\left(\dfrac{r}{2}\right)}\right]^2,$$

$$\text{ARE}(\hat{Q}_n^T, -n\log V; 0.5; r) = 1,$$

and

$$\text{ARE}(\hat{Q}_n^T, -n\log V; \infty; r) = \left[\frac{r}{r+1}\right]^2.$$

These expressions show that the $\text{ARE}(\hat{Q}_n^T, -n\log V)$ decreases as the underlying distribution becomes light-tailed when r = 1. Under the multivariate normal distribution ($\nu = 1$), $\text{ARE}(\hat{Q}_n^T, -n\log V; 1; r)$ is an increasing function of r but with the value of ARE less than 1. Note when r = 1, $\text{ARE}(\hat{Q}_n^T, -n\log V; 0.5; 1)$= 1, $\text{ARE}(\hat{Q}_n^T, -n\log V; 1; 1) = \frac{4}{\pi^2}$, and $\text{ARE}(\hat{Q}_n^T, -n\log V; \infty; 1) = \frac{1}{4}$, which are all identical with the values given in Table 1.1 ($\nu = 0.5$ corresponds to the Laplace, $\nu = 1$ to the normal, and $\nu = \infty$ to the uniform distribution). Since the formula for the ARE's are complex, we use graphs (Figure 3.1 to 3.12) which show the perfomance of one statistic relative to another one in several situations. To simplify the description we consider the case in which $c = 3$ only, and use the Pitman ARE given in Theorem 3.2.1 and Theorem 3.2.2 for various values of $\nu \equiv \nu_1 = \nu_2 = \cdots = \nu_c$ when $d_{1,2}, d_{1,3}, \cdots, d_{c-1,c}$ are assumed to be equal. The height of the surface represents the Pitman ARE at the different dimensions of r_1 and r_2 when the third dimension r_3 is fixed as either one or three. Recall that if $\text{ARE}(T_n, S_n) > (=, <)1$, then T_n is

more(equally, less) efficient. The perfomance of $\hat{Q}_n^T$ relative to $-n\log V$ looks different from situation to situation. $\hat{Q}_n^T$ does very well when $\nu = 0.1$ and $r_3 = 1$ or 3 (Figure 3.1 and 3.2). Thus in very heavy-tailed situation $\hat{Q}_n^T$ is considerably better than the normal theory procedure based on $-n\log V$. There is little difference in ARE($-n\log V$, $\hat{Q}_n^T$) between $r_3 = 1$ and $r_3 = 3$. When $\nu = 0.5$, a moderately heavy-tailed case, $\hat{Q}_n^T$ and $-n\log V$ perform almost equivalently at both $r_3 = 1$ and 3 (Figure 3.3 and 3.4). Here $\hat{Q}_n^T$ has such a tiny performance edge that it is not worth notice. But $-n\log V$ shows better efficiency than $\hat{Q}_n^T$ when $\nu = 1$ (i.e. when the vectors are multivariate normal) at both $r_3 = 1$ and 3 (Figure 3.5 to 3.6). Note as r_3 increases from one to three, it appears that $\hat{Q}_n^T$ becomes more efficient relative to $-n\log V$. This relative perfomance between $\hat{Q}_n^T$ and $-n\log V$ agrees with the result based on two sets of variates in Geiser (1993). Of course, in his case there is no third dimsension to take into account. From all the figures (Figure 3.7 to Figure 3.12), we see that $\hat{Q}_n^T$ is more efficient than $-n\log S^{J_0}$. The test based on $\hat{Q}_n^T$ is more efficient relative to $-n\log S^{J_0}$ when $r_3 = 3$ compared to when $r_3 = 1$.

3.3 Multivariate *t*-distribution Family

The multivariate t-distribution is another elliptically symmetric class of distributions with

$$g_\alpha(w) = \left(1 + \frac{w}{df_\alpha}\right)^{-(df_\alpha + r_\alpha)/2}, K_\alpha = \frac{\Gamma\left(\dfrac{df_\alpha + r_\alpha}{2}\right)}{(\pi df_\alpha)^{r_\alpha/2}\Gamma\left(\dfrac{df_\alpha}{2}\right)},$$

as used in (2.1). With this density $\boldsymbol{X}^{(\alpha)}$ has a multivariate t-distribution centered at zero with $\Sigma_{r_\alpha} = \boldsymbol{I}_{r_\alpha}$ and degrees of freedom df_α. In this case, we will use the notation ($\boldsymbol{X}^{(\alpha)} \sim t(df_\alpha)$). Again we will evaluate ARE($\hat{Q}_n^T, -n\log V$) and ARE($\hat{Q}_n^T, -n\log S^{J_0}$) with different values of df_α. When $df_\alpha = 1$, this corresponds to

the multivariate Cauchy distribution and as $df_\alpha \to \infty$, the distribution approaches the r_α dimensional multivariate normal. We will evaluate the ARE's using the following two lemmas.

Lemma 3.3.1

$$\mathrm{E}_{H_0}\left[\left((R^{(\alpha)})^2\right)^{m_1}\left(1+\frac{(R^{(\alpha)})^2}{df_\alpha}\right)^{m_2}\right] = {df_\alpha}^{m_1}\cdot\frac{B\left(\dfrac{r_\alpha+2m_1}{2},\dfrac{df_\alpha-2m_2-2m_1}{2}\right)}{B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)},$$

provided $df_\alpha > 2(m_1+m_2)$,

where B(·,·) is the beta function.

Proof of Lemma 3.3.1 The density function of S = $(R^{(\alpha)})^2$ is given by

$$q(w) = \frac{1}{df_\alpha B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)}\left(\frac{w}{df_\alpha}\right)^{r_\alpha/2-1}\left(1+\frac{w}{df_\alpha}\right)^{-(df_\alpha+r_\alpha)/2}, \tag{3.4}$$

Then

$$\mathrm{E}\left[\left(\frac{S}{df_\alpha}\right)^{m_1}\left(1+\frac{S}{df_\alpha}\right)^{m_2}\right]$$

$$= \frac{1}{df_\alpha B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)}\int_0^\infty (s/df_k)^{r_k/2-1+m_1}(1+s/df_\alpha)^{-(df_\alpha+r_\alpha)/2+m_2}ds$$

$$= \frac{1}{B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)}\int_0^\infty u^{r_\alpha/2-1+m_1}(1+u)^{-(df_\alpha+r_\alpha)/2+m_2}du$$

$$= \frac{1}{B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)}\int_0^\infty u^{(r_\alpha+2m_1)/2-1}(1+u)^{-((r_\alpha+2m_1)/2+(df_\alpha-2m_1-2m_2)/2)}du$$

$$= \frac{B\left(\dfrac{r_\alpha+2m_1}{2},\dfrac{df_\alpha-2m_1-2m_2}{2}\right)}{B\left(\dfrac{r_\alpha}{2},\dfrac{df_\alpha}{2}\right)},$$

so long as $r_\alpha + 2m_1 > 0$ and $df_\alpha > 2(m_1 + m_2)$.

Lemma 3.3.2

$$\varphi_{\alpha,\beta} = \left(\frac{df_\beta}{df_\alpha}\right)^{1/2} \frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)\Gamma\left(\frac{df_\alpha+1}{2}\right)\Gamma\left(\frac{df_\beta-1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)\Gamma\left(\frac{df_\beta}{2}\right)\Gamma\left(\frac{df_\alpha}{2}\right)},$$

for $1 \leq \alpha < \beta \leq c$.

Proof of Lemma 3.3.2 Since

$$\phi_\alpha(w) = g'_\alpha(w)/g_\alpha(w) = -\frac{df_\alpha + r_\alpha}{2df_\alpha}\left(1 + \frac{w}{df_\alpha}\right)^{-1},$$

$$\begin{aligned}
\varphi_{\alpha,\beta} &= \mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)}\phi_\alpha((R^{(\alpha)})^2)\right] \\
&= -\frac{df_\alpha + r_\alpha}{2df_\alpha}\mathrm{E}_{H_0}\left[R^{(\beta)}\right]\mathrm{E}_{H_0}\left[R^{(\alpha)}\left(1 + \frac{(R^{(\alpha)})^2}{df_\alpha}\right)^{-1}\right] \\
&= -\frac{df_\alpha + r_\alpha}{2df_\alpha}\sqrt{df_\beta}\frac{B\left(\frac{r_\beta+1}{2}, \frac{df_\beta-1}{2}\right)}{B\left(\frac{r_\beta}{2}, \frac{df_\beta}{2}\right)}\sqrt{df_\alpha}\frac{B\left(\frac{r_\alpha+1}{2}, \frac{df_\alpha+1}{2}\right)}{B\left(\frac{r_\alpha}{2}, \frac{df_\alpha}{2}\right)} \\
&= -\frac{df_\alpha + r_\alpha}{2}\left(\frac{df_\beta}{df_\alpha}\right)^{1/2}\frac{B\left(\frac{r_\beta+1}{2}, \frac{df_\beta-1}{2}\right)B\left(\frac{r_\alpha+1}{2}, \frac{df_\alpha+1}{2}\right)}{B\left(\frac{r_\beta}{2}, \frac{df_\beta}{2}\right)B\left(\frac{r_\alpha}{2}, \frac{df_\alpha}{2}\right)} \\
&= \left(\frac{df_\beta}{df_\alpha}\right)^{1/2}\frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)\Gamma\left(\frac{df_\alpha+1}{2}\right)\Gamma\left(\frac{df_\beta-1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)\Gamma\left(\frac{df_\beta}{2}\right)\Gamma\left(\frac{df_\alpha}{2}\right)}.
\end{aligned}$$

Lemmas 3.3.1 and 3.3.2 enable us to state the following Theorem:

Theorem 3.3.1 If $\boldsymbol{X}^{(\alpha)} \sim t(df_\alpha)$, with $df_\alpha > 4$, and we assume that $\boldsymbol{M}_{1,2} = \boldsymbol{M}'_{2,1} = \left((d_{1,2})\right)_{r_1 \times r_2}$, $\boldsymbol{M}_{1,3} = \boldsymbol{M}'_{3,1} = \left((d_{1,3})\right)_{r_1 \times r_3}, \cdots, \boldsymbol{M}_{c-1,c} = \boldsymbol{M}'_{c,c-1} = \left((d_{c-1,c})\right)_{r_{c-1} \times r_c}$,

then

$$\text{ARE}(\hat{Q}_n^T, -n\log V; df_1, df_2, \cdots, df_c) = \frac{\lambda_1^{IV}}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 r_\alpha r_\beta},$$

where

$$\lambda_1^{IV} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \left(\frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)}\right)^2$$

$$\times\left(\frac{\Gamma\left(\frac{df_\alpha+1}{2}\right)\Gamma\left(\frac{df_\beta-1}{2}\right)\left(\frac{df_\beta}{df_\alpha}\right)^{1/2} + \Gamma\left(\frac{df_\alpha-1}{2}\right)\Gamma\left(\frac{df_\beta+1}{2}\right)\left(\frac{df_\alpha}{df_\beta}\right)^{1/2}}{\Gamma\left(\frac{df_\alpha}{2}\right)\Gamma\left(\frac{df_\beta}{2}\right)}\right)^2.$$

Corollary 3.3.1 If $df_1 = df_2 = \cdots = df_c \equiv df$,

$$\text{ARE}(\hat{Q}_n^T, -n\log V; df; r_1, r_2, \cdots, r_c)$$

$$= \frac{1}{\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 r_\alpha r_\beta}\left(\frac{2\Gamma\left(\frac{df+1}{2}\right)\Gamma\left(\frac{df-1}{2}\right)}{\Gamma^2\left(\frac{df}{2}\right)}\right)^2 \times$$

$$\sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2\left(\frac{\Gamma\left(\frac{r_1+1}{2}\right)\Gamma\left(\frac{r_2+1}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\right)^2$$

Corollary 3.3.2 If $df_1 = df_2 = \cdots = df_c \equiv df$ and $r_1 = r_2 = \cdots = r_c \equiv r$ then

$$\text{ARE}(\hat{Q}_n^T, -n\log V; df; r) = \left[\frac{2\Gamma^2\left(\frac{r+1}{2}\right)\Gamma\left(\frac{df+1}{2}\right)\Gamma\left(\frac{df-1}{2}\right)}{r\Gamma^2\left(\frac{r}{2}\right)\Gamma^2\left(\frac{df}{2}\right)}\right]^2, \qquad (3.5)$$

independent of $d_{\alpha,\beta}$'s. This expression is the one obtained by Geiser (1993) under the same conditions for the special case c =2, but it is valid for any c $\geq$ 2. Note that we can rewrite the expression (3.5) as the product of the Pitman ARE under multivariate

normality and a quantity involving only df, which goes to one as $df \to \infty$, i.e.

$$\text{ARE}(\hat{Q}_n^T, -n\log V; df; r) = \left[\frac{\Gamma\left(\frac{df+1}{2}\right)\Gamma\left(\frac{df-1}{2}\right)}{\Gamma^2\left(\frac{df}{2}\right)}\right]^2 \text{ARE}(\hat{Q}_n^T, -n\log V; 1; r). \tag{3.6}$$

Theorem 3.3.2 If $\boldsymbol{X}^{(\alpha)} \sim t(df_\alpha)$, with $df_k > 4$, and we assume that $\boldsymbol{M}_{1,2} = \boldsymbol{M}'_{2,1} = \left((d_{1,2})\right)_{r_1 \times r_2}$, $\boldsymbol{M}_{1,3} = \boldsymbol{M}'_{3,1} = \left((d_{1,3})\right)_{r_1 \times r_3}, \cdots, \boldsymbol{M}_{c-1,c} = \boldsymbol{M}'_{c,c-1} = \left((d_{c-1,c})\right)_{r_{c-1} \times r_c}$, then

$$\text{ARE}(\hat{Q}_n^T, -n\log S^{J_0}) = \frac{\lambda_1^{IV}}{\lambda_3^{IV}},$$

where

$$\lambda_3^{IV} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \left(\frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)}\right)^2 \left(\frac{r_\alpha r_\beta \Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2\Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)}\right)$$

$$\times \left(\frac{\Gamma\left(\frac{df_\alpha+1}{2}\right)\Gamma\left(\frac{df_\beta-1}{2}\right)\left(\frac{df_\beta}{df_\alpha}\right)^{1/2} + \Gamma\left(\frac{df_\alpha-1}{2}\right)\Gamma\left(\frac{df_\beta+1}{2}\right)\left(\frac{df_\alpha}{df_\beta}\right)^{1/2}}{\Gamma\left(\frac{df_\alpha}{2}\right)\Gamma\left(\frac{df_\beta}{2}\right)}\right)^2.$$

Corollary 3.3.3 If $df_1 = df_2 = \cdots = df_c \equiv df$ then

$$\text{ARE}(\hat{Q}_n^T, -n\log S^{J_0}) = \frac{\lambda_1^{V}}{\lambda_3^{V}}, \tag{3.7}$$

where

$$\lambda_1^{V} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \left(\frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)}\right)^2$$

and

$$\lambda_3^{V} = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} d_{\alpha,\beta}^2 \left(\frac{\Gamma\left(\frac{r_\alpha+1}{2}\right)\Gamma\left(\frac{r_\beta+1}{2}\right)}{\Gamma\left(\frac{r_\alpha}{2}\right)\Gamma\left(\frac{r_\beta}{2}\right)}\right)^2 \left(\frac{r_\alpha r_\beta \Gamma^2\left(\frac{r_\alpha}{2}\right)\Gamma^2\left(\frac{r_\beta}{2}\right)}{\pi^2\Gamma^2\left(\frac{r_\alpha+1}{2}\right)\Gamma^2\left(\frac{r_\beta+1}{2}\right)}\right).$$

It is interesting to note that this expression does not depend on the df. Graphs of AREs comparing $\hat{Q}_n^T$ to $-n\log V$ will thus not depend on the df chosen. We include some graphs of ARE($\hat{Q}_n^T$,$-n\log V$) using Theorem 3.3.1 for the various values of $df \equiv df_1 = df_2 = \cdots = df_c$ when $d_{1,2} = d_{1,3} = \cdots = d_{c-1,c}$ and of ARE($\hat{Q}_n^T$, $-n\log S^{J_0}$) in Theorem 3.3.2 when $d_{1,2} = d_{1,3} = \cdots = d_{c-1,c}$. We consider the values of df $= 5$ and 10 only. Since ARE($\hat{Q}_n^T, -n\log V$) exists only under the condition $df > 4$, we exclude df $= 1$. This case will be included in the simulation study found in the next chapter. When df $= 5$ and 10, $-n\log V$ is more efficient than $\hat{Q}_n^T$ (Figure 3.13 to 3.16). It appears that the perfomance of $\hat{Q}_n^T$ at $r_3 = 3$ is better than at $r_3 = 1$ although $-n\log V$ still beats $\hat{Q}_n^T$. However $\hat{Q}_n^T$ beats $-n\log S^{J_0}$ when $r_3 = 1$ or 3 (Figure 3.17 to 3.18).

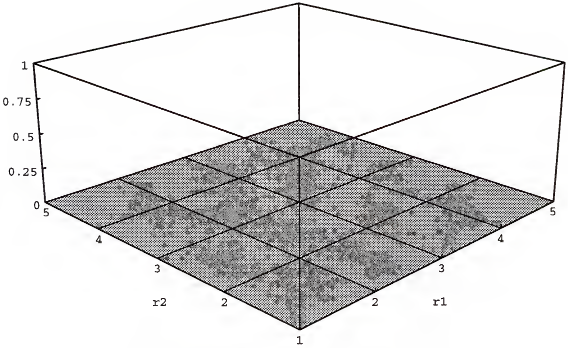

Figure 3.1. ARE($-n \log V$, $\hat{Q}_n^T, \nu = 0.1, r_3 = 1$)

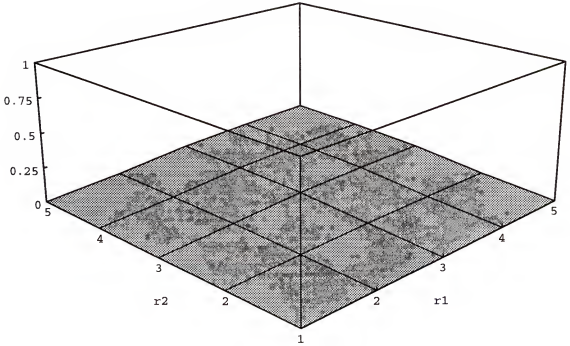

Figure 3.2. ARE($-n \log V$, $\hat{Q}_n^T, \nu = 0.1, r_3 = 3$)

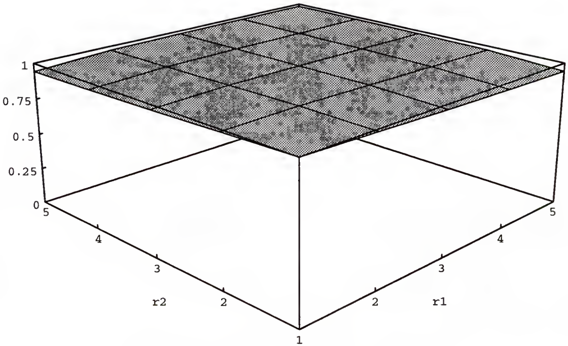

Figure 3.3. ARE($-n \log V$, $\hat{Q}_n^T, \nu = 0.5, r_3 = 1$)

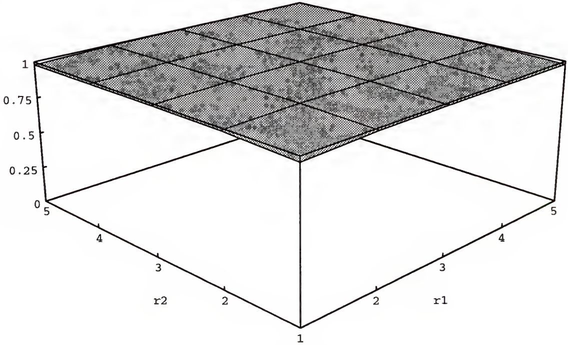

Figure 3.4. ARE($-n \log V$, $\hat{Q}_n^T, \nu = 0.5, r_3 = 3$)

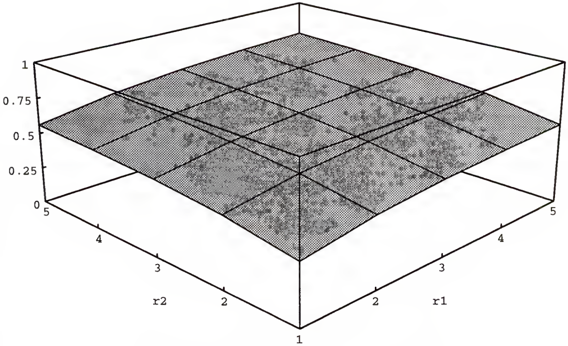

Figure 3.5. ARE($\hat{Q}_n^T$, $-n \log V, \nu = 1, r_3 = 1$)

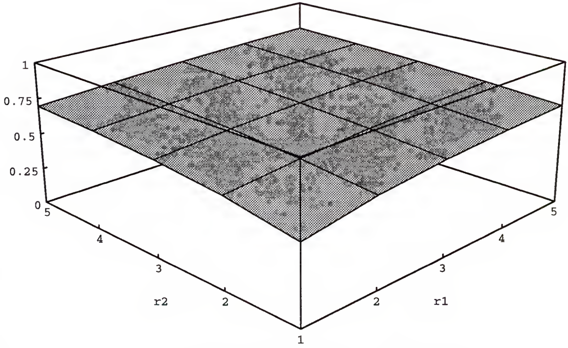

Figure 3.6. ARE($\hat{Q}_n^T$, $-n \log V, \nu = 1, r_3 = 3$)

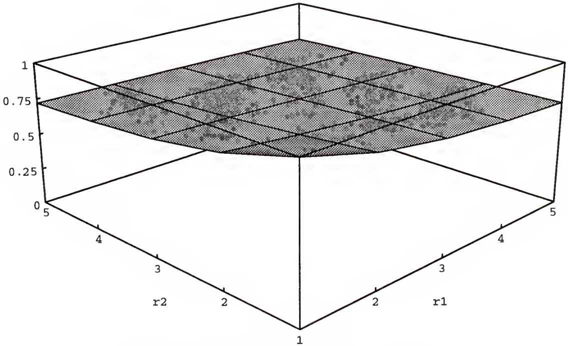

Figure 3.7. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 0.1, r_3 = 1$)

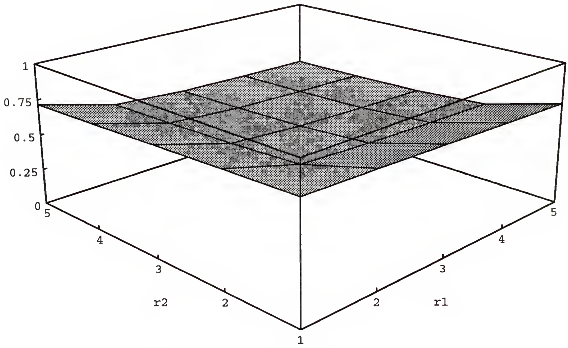

Figure 3.8. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 0.1, r_3 = 3$)

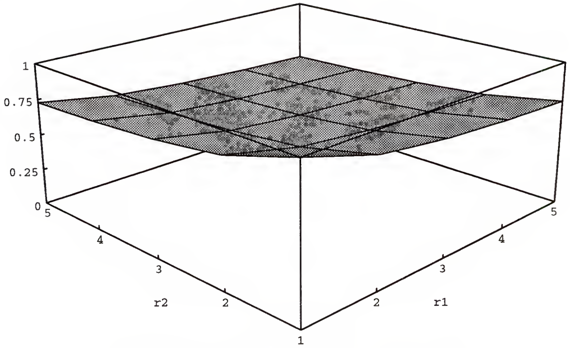

Figure 3.9. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 0.5, r_3 = 1$)

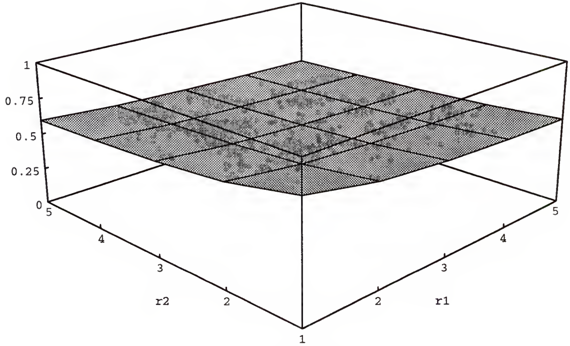

Figure 3.10. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 0.5, r_3 = 3$)

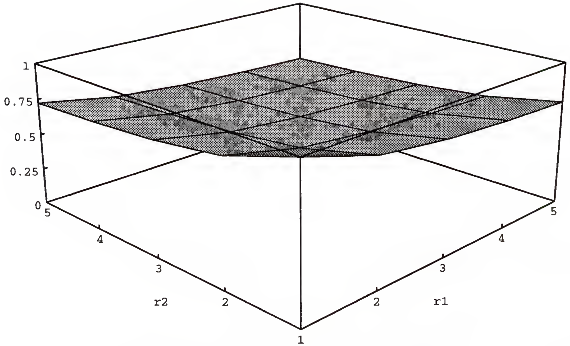

Figure 3.11. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 1, r_3 = 1$)

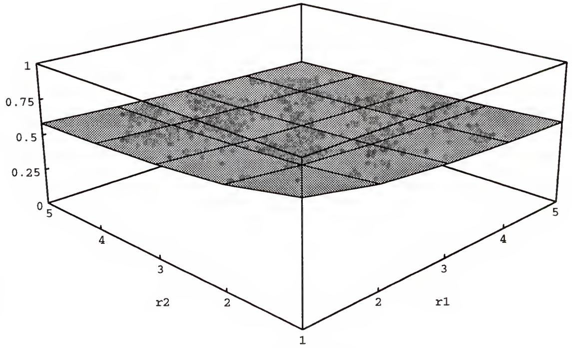

Figure 3.12. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T, \nu = 1, r_3 = 3$)

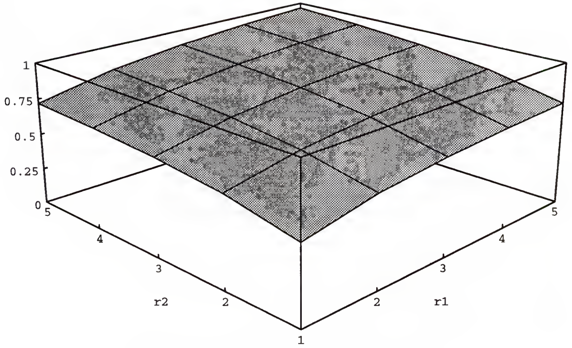

Figure 3.13. ARE($\hat{Q}_n^T$, $-n \log V, df = 5, r_3 = 1$)

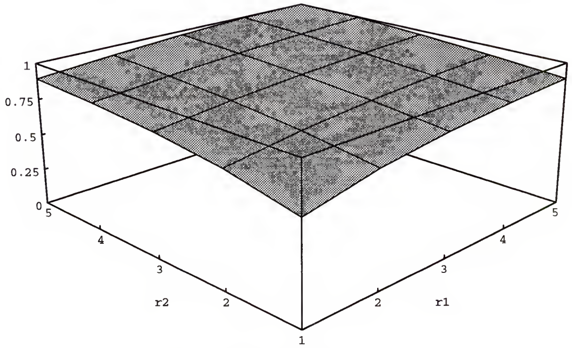

Figure 3.14. ARE($\hat{Q}_n^T$, $-n \log V, df = 5, r_3 = 3$)

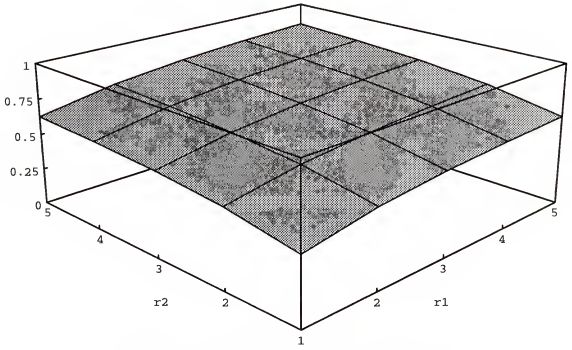

Figure 3.15. ARE($\hat{Q}_n^T$, $-n \log V$, df = 10, $r_3 = 1$)

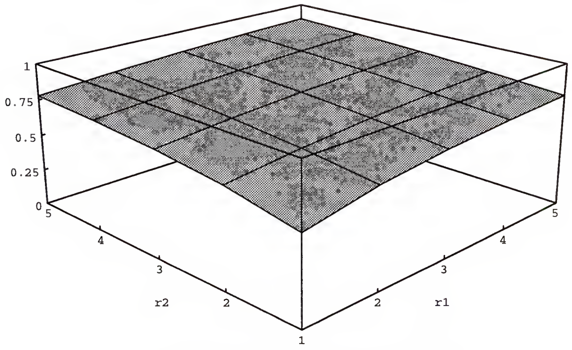

Figure 3.16. ARE($\hat{Q}_n^T$, $-n \log V$, df = 10, $r_3 = 3$)

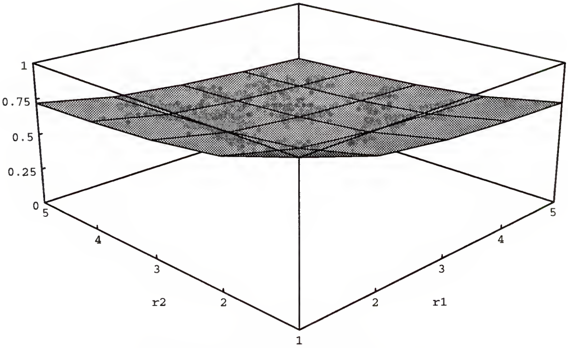

Figure 3.17. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T$, $r_3 = 1$)

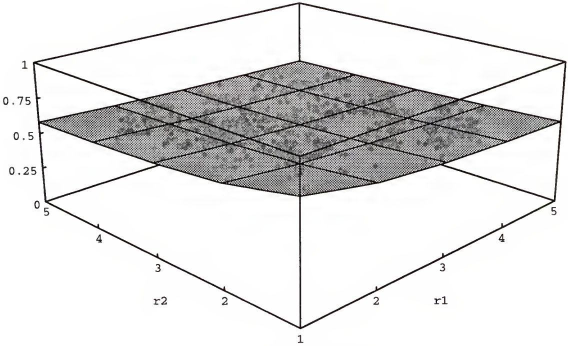

Figure 3.18. ARE($-n \log S^{J_0}$, $\hat{Q}_n^T$, $r_3 = 3$)

CHAPTER 4
INDEPENDENCE PROBLEM : MONTE CARLO STUDY AND EXAMPLE

4.1 Methods and Results

We make additional comparisons of several test procedures using Monte Carlo methods when the number of vectors is three and the sample size is n = 30. The statistics $\hat{Q}_n^T$, $-n\log V$ and $-n\log S^{J_0}$ are included along with two statistics $-n\log V^{1,2}$ $-n\log V^{1,3}$ $-n\log V^{2,3}$ and $-n\log(S^{J_0})^{1,2}$ $-n\log(S^{J_0})^{1,3}$ $-n\log(S^{J_0})^{2,3}$ which are approximations to $-n\log V$ and $-n\log S^{J_0}$, respectively. We use the label L for the statistic $-n\log V$ multiplied by the correction factor

$$1-\frac{3}{2n}-\frac{(r_1+r_2+r_3)^3-\sum_{i=1}^3 r_i^3}{3n((r_1+r_2+r_3)^2-\sum_{i=1}^3 r_i^2)}$$

(Box, 1949). We use the label La for $\sum_{\alpha=1}^2\sum_{\beta=\alpha+1}^3 B_{\alpha,\beta}(-n\log V^{\alpha,\beta})$, where $B_{\alpha,\beta}$ = 1 - $(r_\alpha + r_\beta$ + 3)/(2n) denotes the Bartlett correction factor for the individual term. Consistently, PS and PSa are used to represent $-n\log S^{J_0}$ and $-n\log(S^{J_0})^{1,2}$ $-n\log(S^{J_0})^{1,3}$ $-n\log(S^{J_0})^{2,3}$, respectively.

As estimates of the parameters $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots \boldsymbol{\theta}_c$, we used the Oja median (Oja, 1983) in $\hat{Q}_n^T$ which hence we will call Q1.

Finally we include a statistic Q2, which is asymptotically equivalent to Q1, defined as

$$\text{Q2}=\sum_{\alpha=1}^{2}\sum_{\beta=\alpha+1}^{3}\frac{r_\alpha r_\beta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\boldsymbol{U}}_i^{(\alpha)'}\hat{\boldsymbol{U}}_j^{(\alpha)}\cdot\hat{\boldsymbol{U}}_i^{(\beta)'}\hat{\boldsymbol{U}}_j^{(\beta)},$$

where

$$\hat{U}^{(\alpha)} = \frac{\hat{\Sigma}_{\alpha\alpha}^{-1/2}(X^{(\alpha)} - \hat{\theta}_\alpha)}{\sqrt{(X^{(\alpha)} - \hat{\theta}_\alpha)'\hat{\Sigma}_{\alpha\alpha}^{-1}(X^{(\alpha)} - \hat{\theta}_\alpha)}}$$

and

$$\hat{U}^{(\beta)} = \frac{\hat{\Sigma}_{\beta\beta}^{-1/2}(X^{(\beta)} - \hat{\theta}_\beta)}{\sqrt{(X^{(\beta)} - \hat{\theta}_\beta)'\hat{\Sigma}_{\beta\beta}^{-1}(X^{(\beta)} - \hat{\theta}_\beta)}},$$

and where $\hat{\Sigma}_{\alpha\alpha}$ and $\hat{\Sigma}_{\beta\beta}$ are robust M-estimates of $\Sigma_{\alpha\alpha}$ and $\Sigma_{\beta\beta}$ as described in Randles, Brofitt, Ramberg, and Hogg (1978). Again Oja medians were used to estimate $\theta_1, \theta_2, \cdots \theta_c$.

These six test statistics were compared under the exponential power family and the multivariate t-distributions family. From these distributions, observations are generated using the method by Johnson (1987, p.110). Then model 1 (2.3.1) was used to produce the dependence structure. For comparisons, we considered only the cases where (r_1, r_2, r_3) = (1,1,1), (1,1,2), (1,1,3), (2,2,1), (2,2,2), and (2,2,3) and the underlying distribution types were identical for each set of variables. Specifically, we used the distributions $\nu = 0.1$, 0.5, 1, 10 in the exponential class and df = 1,5 in the multivariate t. The number of repetitions at each case was 2500. In each Monte Carlo simulation, the proportion of times out of 2500 in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution is reported. The asymptotic null distribution $\chi^2_{r_1r_2+r_1r_3+r_2r_3}$ is used to determine the critical value for all the tests. All simulation programs were written in the C programming language. Some already existing routines were taken from archives of software Netlib and Statlib. Also used were parts of the libraries `c/meschach`, `c/cephes` and `ranlib-c`. The final program was compiled using `gcc` (GNU project C compiler v2.4) on a DECstation.

We present the results of the Monte Carlo study in a series of figures and tables. When $\nu = 0.1$ both Q1 and Q2 perform better than the normal theory tests, PS

and PSa in the sense that they have higher power and maintain the nominal size of 0.05 (see Figures 4.1 to 4.6 and Table 4.1). This is true for all cases of (r_1, r_2, r_3) under consideration. For heavy-tailed distributions like the multivariate t with df=1, both Q1 and Q2 are uniformly better than PS and PSa for (r_1, r_2, r_3) = (2,2,1), (2,2,2), and (2,2,3) and maintain the 0.05 significance level (see Figures 4.25 to 4.30 and Table 4.5). Although either PS or PSa is better than Q1 and Q2 at $\Delta = 0.05$ and 0.1 for (r_1, r_2, r_3) = (1,1,1), (1,1,2), and (1,1,3), both of them exceed the 0.05 nominal level. As the distributions become lighter tailed (see Figures 4.7 to 4.24, 4.31 to 4.36 and Tables 4.2,4.3,4.4 and 4.6) the normal theory tests do better than Q1 and Q2. Both PS and PSa beat Q1 and Q2 at various points in the alternatives but their power decreases as the dependency is increased to 0.1 for some cases of (r_1, r_2, r_3) and exceed the 0.05 significance level. In addition, we see that the powers of Q1 and Q2 increase as the dimension changes from (1,1,1) to (1,1,3), (2,2,1) to (2,2,3) and (1,1,1) to (2,2,2) for all distributions. The approximations La and PSa perform better than L and PS respectively for almost all cases of (r_1, r_2, r_3) and Δ.

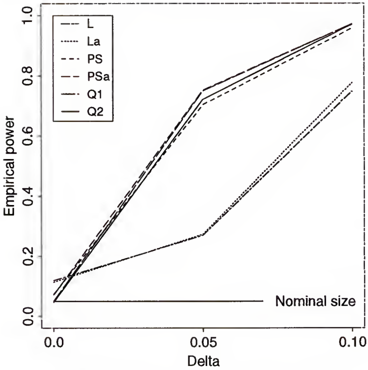

Figure 4.1. $(r_1, r_2, r_3) = (1,1,1)$, $\nu = 0.1$

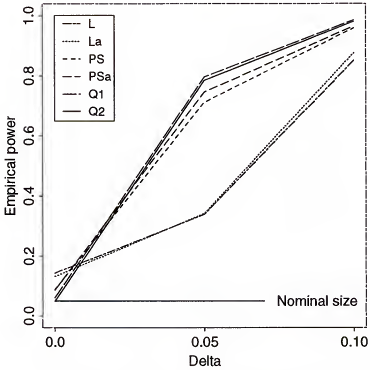

Figure 4.2. $(r_1, r_2, r_3) = (1,1,2)$, $\nu = 0.1$

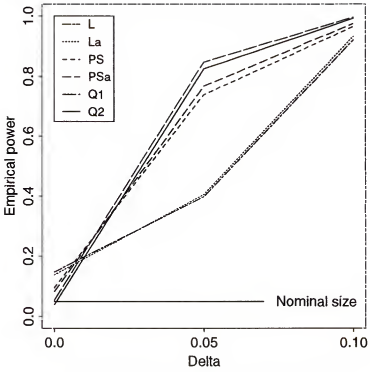

Figure 4.3. $(r_1, r_2, r_3) = (1{,}1{,}3)$, $\nu = 0.1$

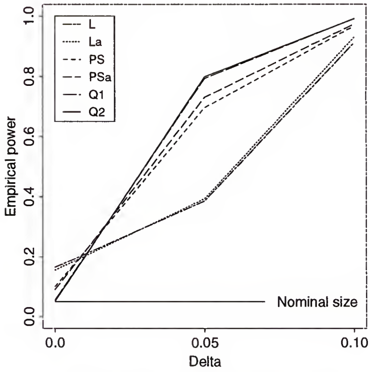

Figure 4.4. $(r_1, r_2, r_3) = (2{,}2{,}1)$, $\nu = 0.1$

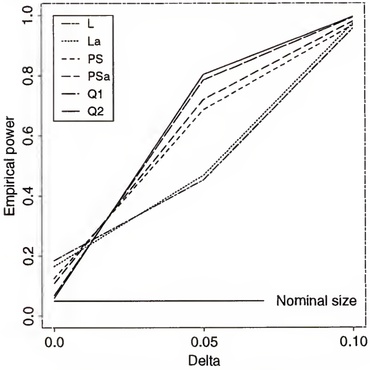

Figure 4.5. $(r_1, r_2, r_3) = (2{,}2{,}2)$, $\nu = 0.1$

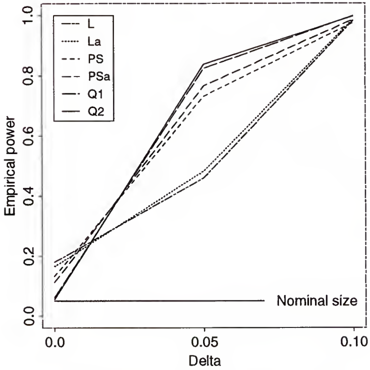

Figure 4.6. $(r_1, r_2, r_3) = (2{,}2{,}3)$, $\nu = 0.1$

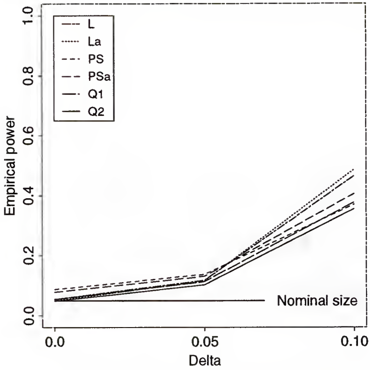

Figure 4.7. $(r_1, r_2, r_3) = (1,1,1)$, $\nu = 0.5$

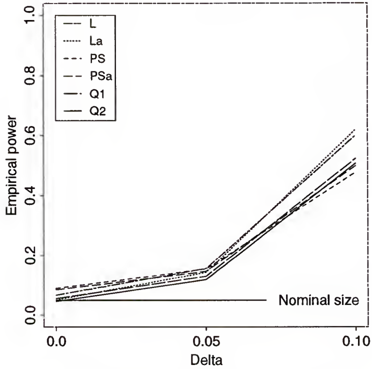

Figure 4.8. $(r_1, r_2, r_3) = (1,1,2)$, $\nu = 0.5$

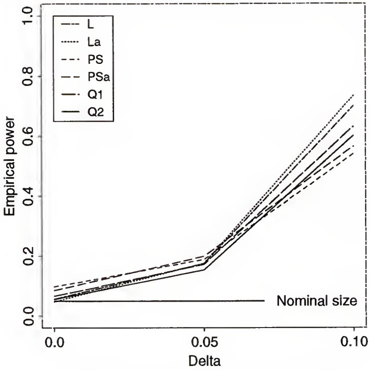

Figure 4.9. $(r_1, r_2, r_3) = (1,1,3)$, $\nu = 0.5$

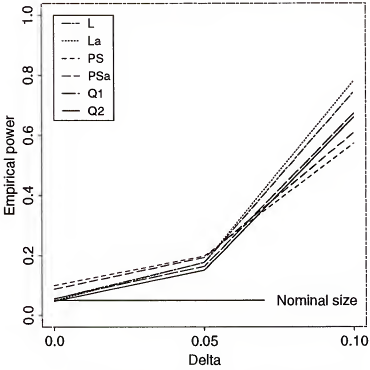

Figure 4.10. $(r_1, r_2, r_3) = (2,2,1)$, $\nu = 0.5$

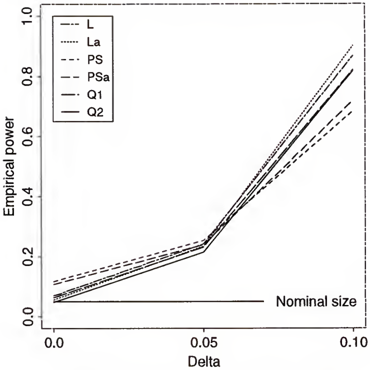

Figure 4.11. $(r_1, r_2, r_3) = (2{,}2{,}2)$, $\nu = 0.5$

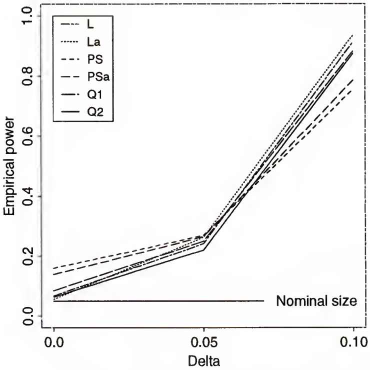

Figure 4.12. $(r_1, r_2, r_3) = (2{,}2{,}3)$, $\nu = 0.5$

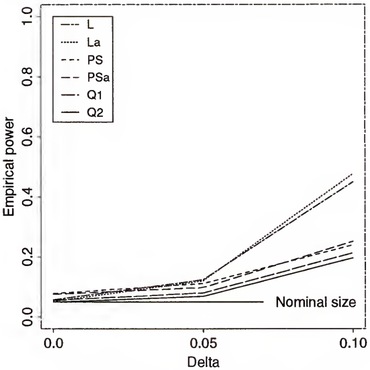

Figure 4.13. $(r_1, r_2, r_3) = (1,1,1)$, $\nu = 1$

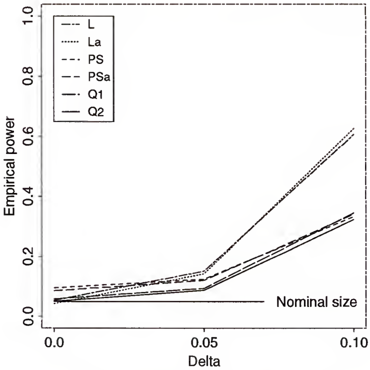

Figure 4.14. $(r_1, r_2, r_3) = (1,1,2)$, $\nu = 1$

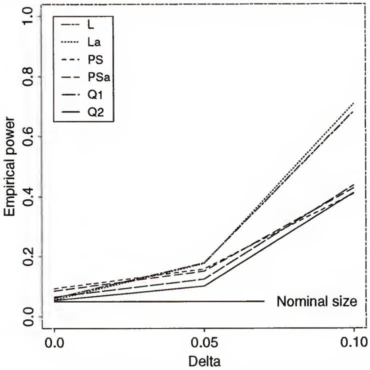

Figure 4.15. $(r_1, r_2, r_3) = (1,1,3)$, $\nu = 1$

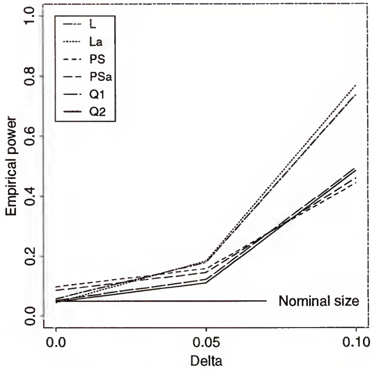

Figure 4.16. $(r_1, r_2, r_3) = (2,2,1)$, $\nu = 1$

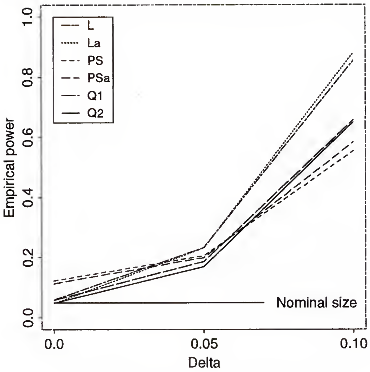

Figure 4.17. $(r_1, r_2, r_3) = (2,2,2)$, $\nu = 1$

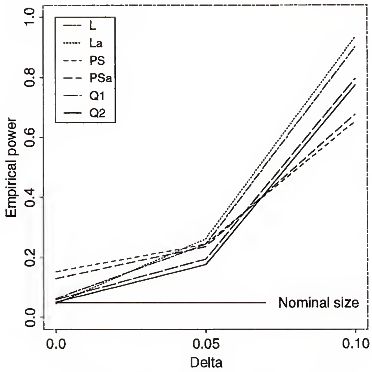

Figure 4.18. $(r_1, r_2, r_3) = (2,2,3)$, $\nu = 1$

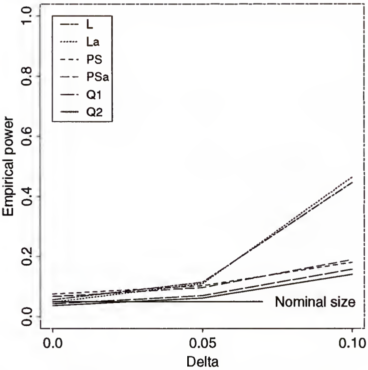

Figure 4.19. $(r_1, r_2, r_3) = (1,1,1)$, $\nu = 10$

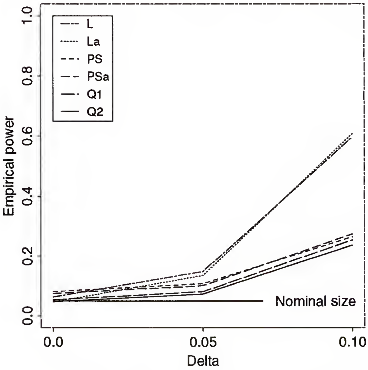

Figure 4.20. $(r_1, r_2, r_3) = (1,1,2)$, $\nu = 10$

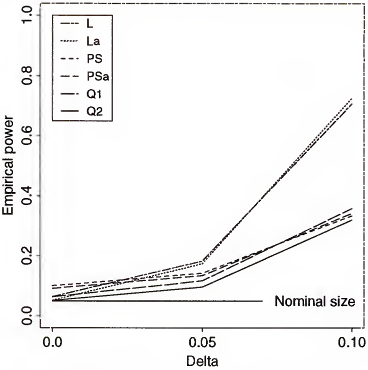

Figure 4.21. $(r_1, r_2, r_3) = (1,1,3)$, $\nu = 10$

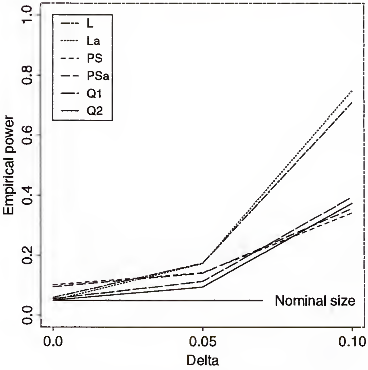

Figure 4.22. $(r_1, r_2, r_3) = (2,2,1)$, $\nu = 10$

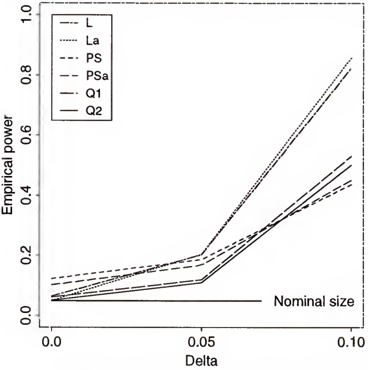

Figure 4.23. $(r_1, r_2, r_3) = (2,2,2)$, $\nu = 10$

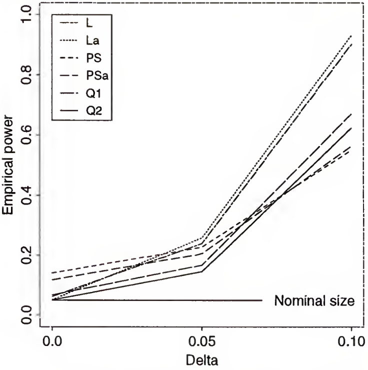

Figure 4.24. $(r_1, r_2, r_3) = (2,2,3)$, $\nu = 10$

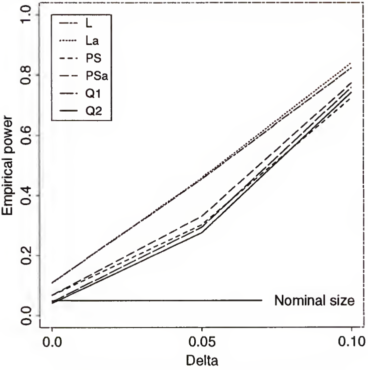

Figure 4.25. $(r_1, r_2, r_3) = (1,1,1)$, df = 1

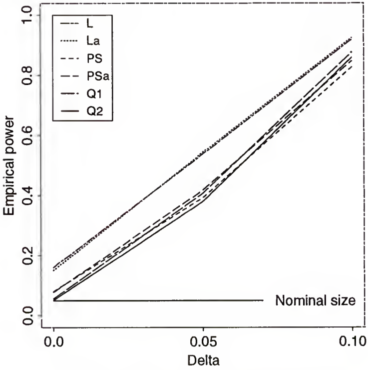

Figure 4.26. $(r_1, r_2, r_3) = (1,1,2)$, df = 1

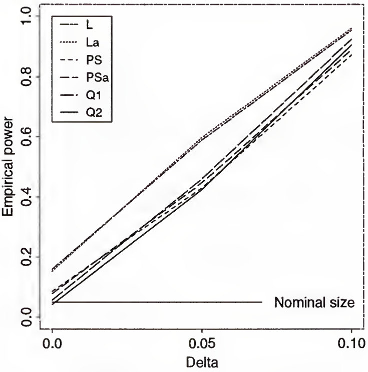

Figure 4.27. $(r_1, r_2, r_3) = (1,1,3)$, df = 1

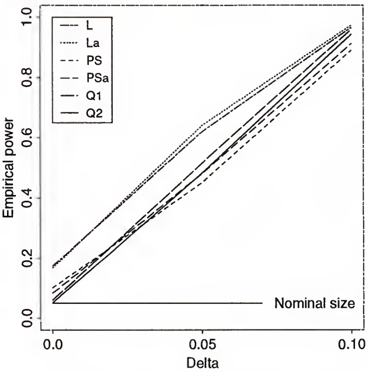

Figure 4.28. $(r_1, r_2, r_3) = (2,2,1)$, df = 1

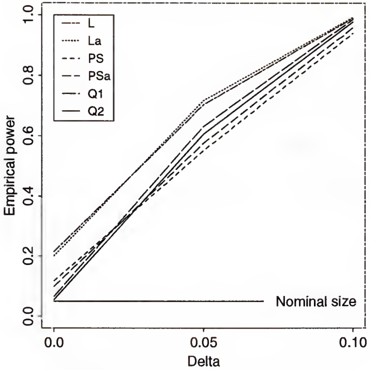

Figure 4.29. (r_1, r_2, r_3) = (2,2,2), df = 1

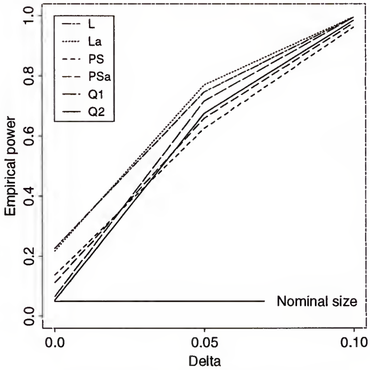

Figure 4.30. (r_1, r_2, r_3) = (2,2,3), df = 1

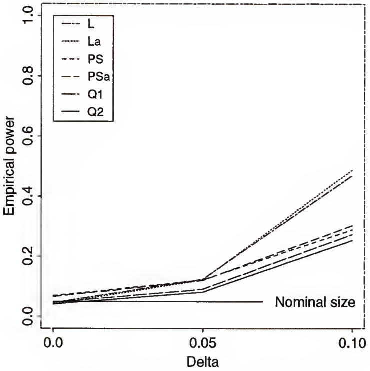

Figure 4.31. $(r_1, r_2, r_3) = (1,1,1)$, df = 5

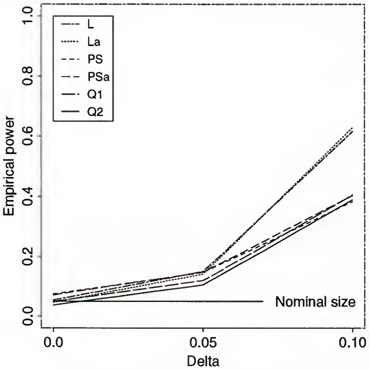

Figure 4.32. $(r_1, r_2, r_3) = (1,1,2)$, df = 5

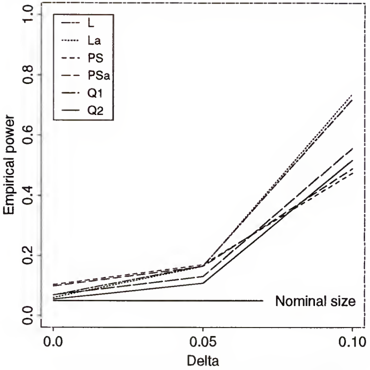

Figure 4.33. $(r_1, r_2, r_3) = (1,1,3)$, df = 5

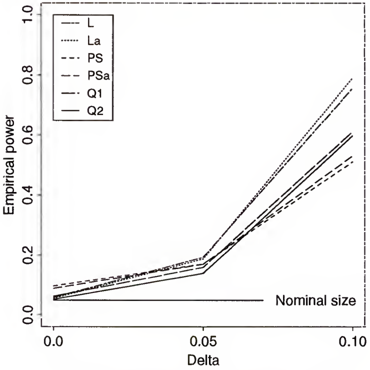

Figure 4.34. $(r_1, r_2, r_3) = (2,2,1)$, df = 5

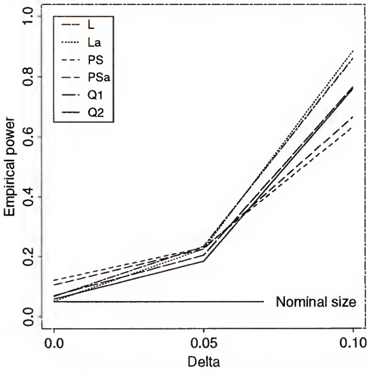

Figure 4.35. $(r_1, r_2, r_3) = (2,2,2)$, df $= 5$

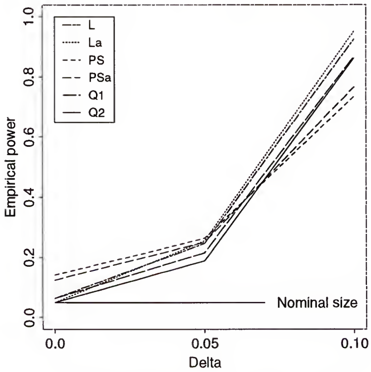

Figure 4.36. $(r_1, r_2, r_3) = (2,2,3)$, df $= 5$

Table 4.1. Estimated Powers for Exponential Power Family
(ν = 0.1, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.1184	0.1128	0.0728	0.0712	0.0500	0.0472
	0.05	0.2700	0.2740	0.7036	0.7524	0.7500	0.7212
	0.10	0.7492	0.7768	0.9572	0.9720	0.9724	0.9708
(1,1,2)	0.00	0.1432	0.1320	0.0880	0.0868	0.0608	0.0484
	0.05	0.3396	0.3424	0.7124	0.7468	0.7964	0.7856
	0.10	0.8520	0.8768	0.9592	0.9656	0.9856	0.9812
(1,1,3)	0.00	0.1476	0.1384	0.0952	0.0824	0.0544	0.0388
	0.05	0.3980	0.4056	0.7384	0.7672	0.8480	0.8252
	0.10	0.9208	0.9336	0.9656	0.9756	0.9960	0.9924
(2,2,1)	0.00	0.1648	0.1548	0.0992	0.0900	0.0552	0.0508
	0.05	0.3864	0.3936	0.6960	0.7304	0.7936	0.7992
	0.10	0.9112	0.9300	0.9672	0.9736	0.9924	0.9912
(2,2,2)	0.00	0.1852	0.1652	0.1244	0.1088	0.0684	0.0600
	0.05	0.4532	0.4684	0.6856	0.7200	0.7856	0.8040
	0.10	0.9568	0.9732	0.9676	0.9800	0.9932	0.9952
(2,2,3)	0.00	0.1800	0.1644	0.1320	0.1120	0.0616	0.0548
	0.05	0.4600	0.4824	0.7312	0.7656	0.8244	0.8380
	0.10	0.9824	0.9864	0.9756	0.9860	0.9984	0.9972

Table 4.2. Estimated Powers for Exponential Power Family
(ν = 0.5, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.0552	0.0504	0.0876	0.0784	0.0544	0.0488
	0.05	0.1180	0.1156	0.1372	0.1308	0.1128	0.1028
	0.10	0.4648	0.4864	0.3704	0.4056	0.3776	0.3552
(1,1,2)	0.00	0.0672	0.0512	0.0896	0.0848	0.0556	0.0460
	0.05	0.1556	0.1428	0.1544	0.1460	0.1300	0.1184
	0.10	0.6000	0.6196	0.4776	0.4996	0.5236	0.5076
(1,1,3)	0.00	0.0564	0.0480	0.0976	0.0856	0.0660	0.0560
	0.05	0.1752	0.1760	0.1880	0.1988	0.1720	0.1536
	0.10	0.7000	0.7340	0.5400	0.5648	0.6312	0.6008
(2,2,1)	0.00	0.0548	0.0492	0.0992	0.0876	0.0560	0.0472
	0.05	0.1768	0.1776	0.1984	0.1940	0.1640	0.1508
	0.10	0.7452	0.7812	0.5728	0.6084	0.6728	0.6600
(2,2,2)	0.00	0.0696	0.0556	0.1172	0.1076	0.0632	0.0480
	0.05	0.2412	0.2344	0.2532	0.2392	0.2308	0.2140
	0.10	0.8676	0.9008	0.6828	0.7176	0.8212	0.8176
(2,2,3)	0.00	0.0672	0.0556	0.1600	0.1388	0.0844	0.0632
	0.05	0.2420	0.2632	0.2696	0.2660	0.2504	0.2216
	0.10	0.9128	0.9368	0.7548	0.7852	0.8824	0.8744

Table 4.3. Estimated Powers for Exponential Power Family
(ν = 1, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.0572	0.0516	0.0780	0.0760	0.0560	0.0492
	0.05	0.1244	0.1204	0.1116	0.0980	0.0796	0.0684
	0.10	0.4504	0.4776	0.2388	0.2512	0.2132	0.1960
(1,1,2)	0.00	0.0544	0.0428	0.0952	0.0860	0.0592	0.0512
	0.05	0.1512	0.1420	0.1240	0.1204	0.0936	0.0872
	0.10	0.6060	0.6252	0.3332	0.3408	0.3448	0.3232
(1,1,3)	0.00	0.0612	0.0556	0.0928	0.0848	0.0656	0.0544
	0.05	0.1784	0.1764	0.1584	0.1512	0.1244	0.1012
	0.10	0.6812	0.7068	0.4092	0.4276	0.4372	0.4120
(2,2,1)	0.00	0.0584	0.0440	0.0980	0.0860	0.0528	0.0480
	0.05	0.1796	0.1840	0.1584	0.1452	0.1220	0.1104
	0.10	0.7360	0.7672	0.4432	0.4596	0.4948	0.4844
(2,2,2)	0.00	0.0588	0.0480	0.1228	0.1128	0.0580	0.0484
	0.05	0.2332	0.2320	0.2060	0.1996	0.1868	0.1696
	0.10	0.8548	0.8808	0.5532	0.5832	0.6580	0.6504
(2,2,3)	0.00	0.0632	0.0464	0.1532	0.1304	0.0600	0.0516
	0.05	0.2456	0.2632	0.2460	0.2372	0.1956	0.1768
	0.10	0.9012	0.9340	0.6540	0.6768	0.7956	0.7740

Table 4.4. Estimated Powers for Exponential Power Family
(ν = 10, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.0568	0.0488	0.0748	0.0660	0.0424	0.0364
	0.05	0.1148	0.1096	0.1036	0.0972	0.0704	0.0624
	0.10	0.4460	0.4636	0.1808	0.1900	0.1572	0.1408
(1,1,2)	0.00	0.0636	0.0468	0.0816	0.0752	0.0540	0.0480
	0.05	0.1484	0.1360	0.1084	0.1012	0.0812	0.0732
	0.10	0.5960	0.6080	0.2652	0.2748	0.2548	0.2372
(1,1,3)	0.00	0.0648	0.0524	0.1008	0.0924	0.0636	0.0516
	0.05	0.1832	0.1748	0.1432	0.1348	0.1176	0.0968
	0.10	0.7072	0.7260	0.3336	0.3420	0.3576	0.3196
(2,2,1)	0.00	0.0596	0.0504	0.1016	0.0952	0.0560	0.0512
	0.05	0.1748	0.1732	0.1424	0.1408	0.1128	0.0948
	0.10	0.7084	0.7484	0.3420	0.3564	0.3952	0.3744
(2,2,2)	0.00	0.0656	0.0512	0.1232	0.1024	0.0620	0.0516
	0.05	0.2024	0.2044	0.1868	0.1684	0.1196	0.1096
	0.10	0.8252	0.8580	0.4360	0.4516	0.5308	0.5020
(2,2,3)	0.00	0.0624	0.0516	0.1408	0.1180	0.0676	0.0512
	0.05	0.2408	0.2596	0.2284	0.2056	0.1672	0.1452
	0.10	0.9016	0.9308	0.5500	0.5644	0.6704	0.6236

Table 4.5. Estimated Powers for Multivariate t-Distribution Family
(df = 1, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.1096	0.1080	0.0672	0.0680	0.0448	0.0400
	0.05	0.4544	0.4584	0.3028	0.3316	0.2952	0.2764
	0.10	0.8232	0.8392	0.7236	0.7736	0.7580	0.7420
(1,1,2)	0.00	0.1604	0.1500	0.0800	0.0772	0.0556	0.0508
	0.05	0.5376	0.5432	0.3928	0.4176	0.4076	0.3800
	0.10	0.9184	0.9240	0.8268	0.8480	0.8764	0.8592
(1,1,3)	0.00	0.1592	0.1528	0.0852	0.0784	0.0572	0.0420
	0.05	0.5892	0.5980	0.4296	0.4468	0.4600	0.4240
	0.10	0.9552	0.9616	0.8724	0.8904	0.9248	0.9060
(2,2,1)	0.00	0.1744	0.1684	0.1008	0.0840	0.0604	0.0520
	0.05	0.6200	0.6404	0.4512	0.4804	0.5164	0.4824
	0.10	0.9660	0.9736	0.8900	0.9116	0.9588	0.9428
(2,2,2)	0.00	0.2160	0.2012	0.1180	0.0988	0.0652	0.0552
	0.05	0.7032	0.7164	0.5512	0.5764	0.6316	0.6056
	0.10	0.9904	0.9936	0.9408	0.9592	0.9864	0.9784
(2,2,3)	0.00	0.2272	0.2184	0.1372	0.1132	0.0660	0.0536
	0.05	0.7464	0.7704	0.6256	0.6592	0.7168	0.6752
	0.10	0.9952	0.9964	0.9624	0.9760	0.9944	0.9852

Table 4.6. Estimated Powers for Multivariate t-Distribution Family (df = 5, n = 30, reps = 2500)

(r_1, r_2, r_3)	Δ	Statistics					
		L	**La**	**PS**	**PSa**	**Q1**	**Q2**
(1,1,1)	0.00	0.0460	0.0412	0.0696	0.0660	0.0472	0.0408
	0.05	0.1244	0.1216	0.1216	0.1200	0.0912	0.0816
	0.10	0.4704	0.4888	0.2896	0.3028	0.2736	0.2540
(1,1,2)	0.00	0.0544	0.0460	0.0740	0.0712	0.0492	0.0372
	0.05	0.1500	0.1416	0.1472	0.1480	0.1192	0.1052
	0.10	0.6156	0.6284	0.3832	0.4032	0.4048	0.3896
(1,1,3)	0.00	0.0684	0.0608	0.1036	0.0988	0.0696	0.0552
	0.05	0.1668	0.1648	0.1704	0.1648	0.1308	0.1084
	0.10	0.7204	0.7376	0.4748	0.4900	0.5568	0.5176
(2,2,1)	0.00	0.0588	0.0556	0.0976	0.0896	0.0632	0.0524
	0.05	0.1936	0.1880	0.1712	0.1704	0.1600	0.1396
	0.10	0.7564	0.7904	0.5112	0.5320	0.6120	0.5980
(2,2,2)	0.00	0.0676	0.0516	0.1208	0.1060	0.0700	0.0592
	0.05	0.2348	0.2260	0.2308	0.2264	0.2052	0.1852
	0.10	0.8608	0.8860	0.6328	0.6660	0.7664	0.7592
(2,2,3)	0.00	0.0640	0.0504	0.1412	0.1252	0.0640	0.0500
	0.05	0.2472	0.2564	0.2640	0.2512	0.2164	0.1888
	0.10	0.9252	0.9508	0.7344	0.7656	0.8676	0.8620

4.2 Example

We will use a real data set to make a final comparison among the six statistics. The data to consider is on Holstein breeder bulls from ABS (American Breeders Service, 1982). The original data consists of 92 observations in the categories of PREDICTED DIFFERENCE, CALVING EASE, GTS and HFA. We take a portion of data, n = 20 observations in three categories of PREDICTED DIFFERENCE, CALVING EASE, and HFA. The variables to consider are DOLLAR, TEST, FAT, CALVING EASE and TYPE. Table 4.7 gives the description of the each variable.

Table 4.7. Description of Variables

DOLLAR	Predicted difference in income from bull's
TEST	Predicted difference in percent butterfat test
FAT	Predicted difference in the amount of fat
CALVE EASE	Percent of difficult births
TYPE	Predicted difference in structural type scores

We are interested in determining if there is any relationship among the three groups of (DOLLAR),(TEST,FAT) and (CALVING EASE,TYPE). A summary of the tests of independence is in Table 4.8. The normal theorey tests, Q1 and Q2 detect the dependence among the groups, while PS and PSa fail to do so. In order to see the influence of a few observations on the result, we miscoded observation 192 for the variable DOLLAR as 092 (see Table 4.8). When we rerun the tests, the normal theory tests become non-significant and the p-value has changed drastically. This is an undesirable property for any test. The statistic Q2 still remains significant and Q1 is close to be significant after replacement. Thus we see the robustness of $\hat{Q}_n^{\,T}$ to slight changes in the data.

Table 4.8. Holstein Data

DOLLAR	TEST	FAT	CALVE EASE	TYPE
218	-0.05	56	12	-0.49
204	0.0	56	16	2.20
200	-0.31	28	8	0.71
192[a]	-0.19	37	19	1.78
188	-0.08	45	13	-0.72
186	-0.11	42	18	1.05
181	-0.27	27	10	1.27
174	0.01	49	19	0.64
169	-0.08	40	12	-0.50
165	0.13	56	14	0.30
161	-0.06	39	23	-0.02
156	-0.16	30	18	0.48
154	-0.15	30	20	1.10
148	-0.07	35	17	1.02
145	0.0	40	10	0.74
140	-0.14	27	11	-0.15
135	-0.05	33	12	1.29
131	0.04	39	7	0.16
128	-0.11	26	15	0.39
98	-0.12	17	8	-0.70

a - in the second computation miscoded as 092

Table 4.9. Statistical Analysis of Holstein Data

			after a is miscoded	
statistic	value	P-value	value	P-value
L	113.3003	0.0000	13.6995	0.0899
La	114.5112	0.0000	12.5819	0.1271
PS	10.5139	0.2308	12.4165	0.1336
PSa	7.9512	0.4382	11.5629	0.1718
Q1	18.4006	0.0184	14.7353	0.0645
Q2	18.6966	0.0166	15.6432	0.0478

CHAPTER 5
MULTIVARIATE MULTI-SAMPLE TEST FOR LOCATION

5.1 Definition of the Test Statistic

A class of multivariate affine-invariant statistics, denoted by $\boldsymbol{Z}_{N,\phi}$, was proposed by Randles and Peters (1993) for testing the difference in location between two elliptically symmetric populations. In this chapter we propose a family of multi-sample affine-invariant multivariate tests for location differences. We assume that $\boldsymbol{X}_i^{(\alpha)} = (\boldsymbol{X}_{i1}^{(\alpha)},\ldots, \boldsymbol{X}_{ip}^{(\alpha)})'$, $i = 1,\ldots,n_\alpha$, denotes a random sample from a p-variate continuous population with an elliptically symmetric distribution with density given in (2.1). The p-dimensional vector $\boldsymbol{\theta}_\alpha$ is the point of symmetry. We assume that the c samples corresponding to $\alpha = 1,\ldots,c$ are independent of one another and that the populations differ only by their location parameters, i.e. $g_\alpha \equiv g$ and $\boldsymbol{\Sigma}_\alpha \equiv \boldsymbol{\Sigma}$. We are interested in testing $H_0 : \boldsymbol{\theta}_1 = \cdots = \boldsymbol{\theta}_c = \boldsymbol{\theta}$ against the alternative that the location parameters $\boldsymbol{\theta}_1, \ldots ,\boldsymbol{\theta}_c$ are not all equal. The family of statistics we propose is built on the two-sample procedures given in Randles and Peters (1993). We make the extension to c-sample setting by summing all possible pairs of two-sample statistics. Thus the

class of statistics we propose is given by

$$\begin{aligned}
\boldsymbol{W}_{N,\phi} &= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \boldsymbol{Z}_{N,\phi}^{\alpha,\beta} \\
&= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \frac{n_\alpha n_\beta}{NE(\phi^2)} \left\{ \frac{1}{n_\alpha^2} \sum_{i=1}^{n_\alpha} \sum_{j=1}^{n_\alpha} \cos\left(\pi \hat{p}_\alpha(i,j;\hat{\boldsymbol{\theta}})\right) \phi\left(\frac{R_{\alpha,i}}{N}\right) \phi\left(\frac{R_{\alpha,j}}{N}\right) \right. \\
&\quad + \frac{1}{n_\beta^2} \sum_{i=1}^{n_\beta} \sum_{j=1}^{n_\beta} \cos\left(\pi \hat{p}_\beta(i,j;\hat{\boldsymbol{\theta}})\right) \phi\left(\frac{R_{\beta,i}}{N}\right) \phi\left(\frac{R_{\beta,j}}{N}\right) \\
&\quad \left. - \frac{2}{n_\alpha n_\beta} \sum_{i=1}^{n_\alpha} \sum_{j=1}^{n_\beta} \cos\left(\pi \hat{p}_{\alpha,\beta}(i,j;\hat{\boldsymbol{\theta}})\right) \phi\left(\frac{R_{\alpha,i}}{N}\right) \phi\left(\frac{R_{\beta,j}}{N}\right) \right\},
\end{aligned}$$

where $\boldsymbol{Z}_{N,\phi}^{\alpha,\beta}$ compares the αth and βth samples' locations. Note that this term is indexed by the total number of observations N = $n_1 + n_2 + \cdots + n_c$. Here ϕ is a nondecreasing score function defined on (0,1) where ϕ must satisfy $0 < E(\phi^2) = \int_0^1 \phi^2(u) < \infty$. As score functions ϕ, we use two different functions $\phi_1(u) = 1$ and $\phi_2(u) =$ u. In the formula above, $\hat{\boldsymbol{\theta}}$ is an affine equivariant estimator of the common location parameter $\boldsymbol{\theta}$ if H_0 is true. Randles and Peters (1990) used the sample mean of the N observations for $\hat{\boldsymbol{\theta}}$. For our case we use another affine-invariant estimator, the Oja median (Oja,1983). In the above expression, $R_{\alpha,i}$ denotes the rank of distance $D_{\alpha,i}$ among all N distances $D_{1,1},\ldots,\ D_{1,n_1},\ldots,D_{c,1},\ldots,D_{c,n_c}$, where

$$D_{\alpha,i} = (\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})'\hat{\Sigma}^{-1}(\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})$$

and

$$\hat{\Sigma} = \frac{1}{N} \sum_{\alpha=1}^{c} \sum_{1=1}^{n_\alpha} (\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})(\boldsymbol{X}_i^{(\alpha)} - \hat{\boldsymbol{\theta}})'$$

The term $\hat{p}_\alpha(i,j;\hat{\boldsymbol{\theta}})$ is the sample proportion of hyperplanes formed by $\hat{\boldsymbol{\theta}}$ and p-1 of the observations $\boldsymbol{X}_{i'}^{(\alpha)}$ ($i' =$ 1,...,n_α, but $i \neq i' \neq$ j) such that $\boldsymbol{X}_i^{(\alpha)}$ and $\boldsymbol{X}_j^{(\alpha)}$ are on opposite sides of the hyperplane formed. The term $\hat{p}_\beta(i,j;\hat{\boldsymbol{\theta}})$ is similary defined. Here

$\boldsymbol{X}_i^{(\alpha)}$ ($\boldsymbol{X}_i^{(\beta)}$) is the ith observation from the αth (βth) sample. The term $\hat{p}_{\alpha,\beta}(i,j;\hat{\boldsymbol{\theta}})$ represents the sample proportion of hyperplanes formed by $\hat{\boldsymbol{\theta}}$ and p-1 of the $n_\alpha + n_\beta - 2$ observations ($\boldsymbol{X}_{i'}^{(\alpha)}$, i' = 1,...,n_α with $i' \neq$ i and $\boldsymbol{X}_{j'}^{(\beta)}$, j' = 1,...,n_β with $j' \neq$ j) such that $\boldsymbol{X}_i^{(\alpha)}$ (coming from the αth sample) and $\boldsymbol{X}_j^{(\beta)}$ (coming from the βth sample) are on opposite sides of the hyperplane formed.

5.2 Asymptotic Null Distribution of $\boldsymbol{W}_{N,\phi}$

In this section we will find the asymptotic null distribution of $\boldsymbol{W}_{N,\phi}$ under the class of elliptically symmetric distributions. Let $\boldsymbol{X}_i^{(\alpha)}$, $\alpha = 1,\ldots,c$, $i = 1,\ldots,n_\alpha$, be independent random samples from some elliptically symmetric distributions with dispersion matrix $\boldsymbol{\Sigma}$ and a location parameter $\boldsymbol{\theta}$. We assume that both $\hat{\Sigma}$ - Σ = $O_p(N^{-1/2})$ and $\hat{\boldsymbol{\theta}}$ - $\boldsymbol{\theta}$ = $O_p(N^{-1/2})$ and that $\lim_{N\to\infty} \frac{n_\alpha}{N} = \lambda_\alpha$ with $0 < \lambda_\alpha < 1$, and $\lambda_1 + \lambda_2 + \cdots + \lambda_c = 1$. Because the statistics $\boldsymbol{Z}_{N,\phi}^{\alpha,\beta}$ are all affine invariant, we assume without loss of generality that $\boldsymbol{\Sigma} = \boldsymbol{I}(p \times p)$ and that $\boldsymbol{\theta}$ is the origin. Similarly to the development in section 2.2, we use the approximating statistic which has the same asymptotic null distribution as $\boldsymbol{W}_{N,\phi}$. Randles and Peters (1993) showed that approximating quantity for $\boldsymbol{Z}_{N,\phi}^{\alpha,\beta}$ in the two sample case introduced is:

$$\begin{aligned}
\boldsymbol{Z}_{N,\phi}^{*,\alpha,\beta} = \frac{n_\alpha n_\beta}{NE(\phi^2)}\Bigg\{ & \frac{1}{n_\alpha^2}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\alpha} \cos\left(\pi p_\alpha(i,j;\hat{\boldsymbol{\theta}})\right)\phi\left(H(R_{\alpha,i}^*)\right)\phi\left(H(R_{\alpha,j}^*)\right) \\
& + \frac{1}{n_\beta^2}\sum_{i=1}^{n_\beta}\sum_{j=1}^{n_\beta} \cos\left(\pi p_\beta(i,j;\hat{\boldsymbol{\theta}})\right)\phi\left(H(R_{\beta,i}^*)\right)\phi\left(H(R_{\beta,j}^*)\right) \\
& - \frac{2}{n_\alpha n_\beta}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta} \cos\left(\pi p_{\alpha,\beta}(i,j;\hat{\boldsymbol{\theta}})\right)\phi\left(H(R_{\alpha,i}^*)\right)\phi\left(H(R_{\beta,j}^*)\right)\Bigg\}
\end{aligned}$$

where $R_{\alpha,i}^* = \boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{X}_i^{(\alpha)}$, $R_{\beta,i}^* = \boldsymbol{X}_i^{(\beta)\prime}\boldsymbol{X}_i^{(\beta)}$, H is the distribution function of $R_{\alpha,i}^*$ (and $R_{\beta,i}^*$) under H_0, and $p_\alpha(i,j;\hat{\boldsymbol{\theta}})$ is the angle in radians between $\boldsymbol{X}_i^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{X}_j^{(\alpha)}$ - $\hat{\boldsymbol{\theta}}$

(similarly for $p_\beta(i,j;\hat{\boldsymbol{\theta}})$ and $p_{\alpha,\beta}(i,j;\hat{\boldsymbol{\theta}})$). Therefore our approximating quantity for $\boldsymbol{W}_{N,\phi}$ is given by:

$$\boldsymbol{W}^*_{N,\phi} = \boldsymbol{Z}^{*,1,2}_{N,\phi} + \boldsymbol{Z}^{*,1,3}_{N,\phi} + \cdots + \boldsymbol{Z}^{*,c-1,c}_{N,\phi} \tag{5.1}$$

Randles and Peters (1993) showed that $\boldsymbol{Z}^{\alpha,\beta}_{N,\phi}$ - $\boldsymbol{Z}^{*,\alpha,\beta}_{N,\phi} = o_p(1)$ under H_0, following an similar approach to that in the Peters and Randles (1990,1991). It follows that $\boldsymbol{W}_{N,\phi}$ and $\boldsymbol{W}^*_{N,\phi}$ have the same asymptotic null distribution. Noting that $\cos(\pi p_\alpha(i,j;\hat{\boldsymbol{\theta}}))$ $= \boldsymbol{U}^{(\alpha)\prime}_i \boldsymbol{U}^{(\alpha)}_i$ (similarly $\cos(\pi p_\beta(i,j;\hat{\boldsymbol{\theta}})) = \boldsymbol{U}^{(\beta)\prime}_i \boldsymbol{U}^{(\beta)}_i$) as seen in section 2.2. we can rewrite $\boldsymbol{Z}^{*,\alpha,\beta}_{N,\phi}$ as $\boldsymbol{S}^{\alpha,\beta\,\prime}_{N,\phi} \boldsymbol{S}^{\alpha,\beta}_{N,\phi}$ where

$$\boldsymbol{S}^{\alpha,\beta}_{N,\phi} = \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}} \left\{ \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} \boldsymbol{U}^{(\alpha)}_i \phi\big(H(R^*_{\alpha,i})\big) - \frac{1}{n_\beta} \sum_{j=1}^{n_\beta} \boldsymbol{U}^{(\beta)}_j \phi\big(H(R^*_{\beta,j})\big) \right\} (-1)^{k_{\alpha,\beta}}.$$

Here the constant $(-1)^{k_{\alpha,\beta}}$ is used to show that the two summations inside the curly bracket are switched when the value of $k_{\alpha,\beta}$ is 1. Here $k_{\alpha,\beta}$ is given by

$$k_{\alpha,\beta} = \begin{pmatrix} 1 & \text{if } c \text{ is odd and } |\alpha - \beta| \geq \frac{c+1}{2} \\ 1 & \text{if } c \text{ is even and } |\alpha - \beta| \geq \frac{c}{2} + 1 \\ 0 & \text{otherwise} \end{pmatrix}$$

For instance, when $c = 4$, there are a total of six pairs of (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) that can be formed from the set of numbers (1,2,3,4). For each pair we get the corresponding $\boldsymbol{S}^{\alpha,\beta}_{N,\phi}$ defined above. But since c is even and $|1-4| = 4/2 + 1$ $= 3$ for the pair of (1,4), we have

$$\boldsymbol{S}^{1,4}_{N,\phi} = \frac{\sqrt{n_1 n_4 p}}{\sqrt{NE(\phi^2)}} \left\{ \frac{1}{n_4} \sum_{j=1}^{n_4} \boldsymbol{U}^{(4)}_j \phi\big(H(R^*_{4,j})\big) - \frac{1}{n_1} \sum_{i=1}^{n_1} \boldsymbol{U}^{(1)}_i \phi\big(H(R^*_{1,i})\big) \right\}$$

Thus

$$\begin{aligned}
\boldsymbol{W}^*_{N,\phi} &= \boldsymbol{S}^{1,2\,\prime}_{N,\phi}\boldsymbol{S}^{1,2}_{N,\phi} + \boldsymbol{S}^{1,3\,\prime}_{N,\phi}\boldsymbol{S}^{1,3}_{N,\phi} + \cdots + \boldsymbol{S}^{c-1,c\prime}_{N,\phi}\boldsymbol{S}^{c-1,c}_{N,\phi} \\
&= \left(\boldsymbol{S}^{1,2\,\prime}_{N,\phi}, \boldsymbol{S}^{1,3\,\prime}_{N,\phi}, \cdots, \boldsymbol{S}^{c-1,c\prime}_{N,\phi}\right)\begin{pmatrix} \boldsymbol{S}^{1,2}_{N,\phi} \\ \boldsymbol{S}^{1,3}_{N,\phi} \\ \vdots \\ \boldsymbol{S}^{c-1,c}_{N,\phi} \end{pmatrix} \\
&\equiv \boldsymbol{S}_{N,\phi}{}'\boldsymbol{S}_{N,\phi}
\end{aligned}$$

We are now prepared to state the following theorem.

Theorem 5.2.1 Under H_0, $\boldsymbol{W}_{N,\phi} \xrightarrow{d} \chi^2_{(c-1)p}$

Proof of Theorem 5.2.1 Let $\boldsymbol{B} = (\boldsymbol{B}'_{1,2}, \boldsymbol{B}'_{2,3}, \ldots, \boldsymbol{B}'_{c-1,c})'$ where $\boldsymbol{B}_{1,2} = \{b_t^{1,2}\}_{p\times 1}$, $\boldsymbol{B}_{2,3} = \{b_t^{2,3}\}_{p\times 1}$, $\cdots$, $\boldsymbol{B}_{c-1,c} = \{b_t^{c-1,c}\}_{p\times 1}$, are arbitary nonzero fixed vectors. Write $\boldsymbol{S}^{1,2}_{N,\phi} = \{s_t^{1,2}\}_{p\times 1}$, $\boldsymbol{S}^{2,3}_{N,\phi} = \{s_t^{2,3}\}_{p\times 1}, \cdots$, $\boldsymbol{S}^{c-1,c}_{N,\phi} = \{s_t^{c-1,c}\}_{p\times 1}$, where

$$\begin{aligned}
s_t^{\alpha,\beta} &= \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\left(\frac{1}{n_\alpha}\sum_{i=1}^{n_\alpha} U_{i,t}^{(\alpha)}\phi\Big(H(R^*_{\alpha,i})\Big) - \frac{1}{n_\beta}\sum_{j=1}^{n_\beta} U_{j,t}^{(\beta)}\phi\Big(H(R^*_{\beta,j})\Big)\right)(-1)^{k_{\alpha,\beta}} \\
&= \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\frac{1}{n_\alpha n_\beta}\left[\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta}\left(U_{i,t}^{(\alpha)}\phi\Big(H(R^*_{\alpha,i})\Big) - U_{j,t}^{(\beta)}\phi\Big(H(R^*_{\beta,j})\Big)\right)\right](-1)^{k_{\alpha,\beta}}.
\end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{B}'\boldsymbol{S}_{N,\phi} &= \boldsymbol{B}'_{1,2}\boldsymbol{S}^{1,2}_{N,\phi} + \boldsymbol{B}'_{1,3}\boldsymbol{S}^{1,3}_{N,\phi} + \cdots + \boldsymbol{B}'_{c-1,c}\boldsymbol{S}^{c-1,c}_{N,\phi} \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}\sum_{t=1}^{p} b_t^{\alpha,\beta} s_t^{\alpha,\beta} \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\frac{1}{n_\alpha n_\beta}\sum_{t=1}^{p}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta} b_t^{\alpha\beta}\Big(U_{i,t}^{(\alpha)}\phi\big(H(R^*_{\alpha,i})\big) \\
&\qquad\qquad - U_{j,t}^{(\beta)}\phi\big(H(R^*_{\beta,j})\big)\Big)(-1)^{k_{\alpha,\beta}} \\
&= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c} \frac{\sqrt{n_\alpha n_\beta p}}{\sqrt{NE(\phi^2)}}\frac{1}{n_\alpha n_\beta}\sum_{i=1}^{n_\alpha}\sum_{j=1}^{n_\beta} \boldsymbol{B}'_{\alpha,\beta}\Big(\boldsymbol{U}_i^{(\alpha)}\phi\big(H(R^*_{\alpha,i})\big) \\
&\qquad\qquad - \boldsymbol{U}_j^{(\beta)}\phi\big(H(R^*_{\beta,j})\big)\Big)(-1)^{k_{\alpha,\beta}} \\
&= \frac{1}{n_1}\sum_{i=1}^{n_1}\left\{\frac{\sqrt{n_1 n_2 p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,2} + \cdots - \frac{\sqrt{n_1 n_c p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,c}\right\}\Big(\boldsymbol{U}_i^{(1)}\phi\big(H(R^*_{1,i})\big)\Big) \\
&\quad + \frac{1}{n_2}\sum_{i=1}^{n_2}\left\{\frac{\sqrt{n_2 n_3 p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{2,3} + \cdots - \frac{\sqrt{n_1 n_2 p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,2}\right\}\Big(\boldsymbol{U}_i^{(2)}\phi\big(H(R^*_{2,i})\big)\Big) \\
&\quad + \cdots \\
&\quad + \frac{1}{n_{c-1}}\sum_{i=1}^{n_{c-1}}\left\{\frac{\sqrt{n_{c-1} n_c p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{c-1,c} + \cdots - \frac{\sqrt{n_{c-2} n_{c-1} p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{c-2,c-1}\right\} \\
&\qquad\qquad \cdot\Big(\boldsymbol{U}_i^{(c-1)}\phi\big(H(R^*_{c-1,i})\big)\Big) \\
&\quad + \frac{1}{n_c}\sum_{i=1}^{n_c}\left\{\frac{\sqrt{n_1 n_c p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,c} + \cdots - \frac{\sqrt{n_{c-1} n_c p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{c-1,c}\right\}\Big(\boldsymbol{U}_i^{(c)}\phi\big(H(R^*_{c,i})\big)\Big).
\end{aligned}$$

Here we see that the summands in each term are iid random variables so that we can establish an asymptotic normal distribution for each term by the central limit theorem. The first term converges to $\mathrm{N}\Big(\mathbf{0}, (\sqrt{\lambda_2}\boldsymbol{B}'_{1,2} + \cdots - \sqrt{\lambda_c}\boldsymbol{B}'_{1,c})\ (\sqrt{\lambda_2}\boldsymbol{B}'_{1,2} +$

$\cdots - \sqrt{\lambda_c}\boldsymbol{B}'_{1,c})'\Big)$, the second one to $\mathrm{N}\Big(\boldsymbol{0}, (\sqrt{\lambda_3}\boldsymbol{B}'_{2,3} + \cdots - \sqrt{\lambda_1}\boldsymbol{B}_{1,2})\ (\sqrt{\lambda_3}\boldsymbol{B}'_{2,3} + \cdots - \sqrt{\lambda_1}\boldsymbol{B}_{1,2})'\Big)$, and the cth one to $\mathrm{N}\Big(\boldsymbol{0}, (\sqrt{\lambda_1}\boldsymbol{B}'_{1,c} + \cdots - \sqrt{\lambda_{c-1}}\boldsymbol{B}'_{c-1,c})(\sqrt{\lambda_1}\boldsymbol{B}'_{1,c} + \cdots - \sqrt{\lambda_{c-1}}\boldsymbol{B}'_{c-1,c})'\Big)$, etc, where $\lim_{N\to\infty} \frac{n_i}{N} = \lambda_i$, i = 1,...,$c$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_c = 1$. Also, since the terms are independent of one another, we can obtain the asymptotic normality of $\boldsymbol{B}'\boldsymbol{S}_{N,\phi}$ via multi-sample version of central limit theorem. So that

$$\boldsymbol{B}'\boldsymbol{S}_{N,\phi} \xrightarrow{d} N(\boldsymbol{0}, \Psi),$$

where

$$\Psi = \left(\begin{pmatrix} \sqrt{\lambda_2}I_p & \cdots & -\sqrt{\lambda_c}I_p \\ -\sqrt{\lambda_1}I_p & \sqrt{\lambda_3}I_p & \cdots \\ \vdots & & \\ \cdots & \sqrt{\lambda_{c-1}} & \sqrt{\lambda_1}I_p \end{pmatrix} \boldsymbol{B}\right)' \left(\begin{pmatrix} \sqrt{\lambda_2}I_p & \cdots & -\sqrt{\lambda_c}I_p \\ -\sqrt{\lambda_1}I_p & \sqrt{\lambda_3}I_p & \cdots \\ \vdots & & \\ \cdots & \sqrt{\lambda_{c-1}}I_p & \sqrt{\lambda_1}I_p \end{pmatrix} \boldsymbol{B}\right),$$

hence

$$\boldsymbol{S}_{N,\phi} \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{V}) \tag{5.2}$$

where $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$ is $\frac{(c-1)cp}{2} \times \frac{(c-1)cp}{2}$ var-covaiance matrix and

$$\boldsymbol{A} = \begin{pmatrix} \sqrt{\lambda_2}I_p & \cdots & \cdots & -\sqrt{\lambda_c}I_p \\ -\sqrt{\lambda_1}I_p & \sqrt{\lambda_3}I_p & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \sqrt{\lambda_1}I_p \end{pmatrix}_{cp\times\frac{(c-1)cp}{2}}.$$

For example when $c = 4$, the expression for $\boldsymbol{B}'\boldsymbol{S}_{N,\phi}$ reduces to

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left\{\frac{\sqrt{n_1n_2p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,2}+\frac{\sqrt{n_1n_3p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,3}-\frac{\sqrt{n_1n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,4}\right\}\left(\boldsymbol{U}_i^{(1)}\phi\big(H(R^*_{1,i})\big)\right)$$

$$+\frac{1}{n_2}\sum_{i=1}^{n_2}\left\{\frac{\sqrt{n_2n_3p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{2,3}+\frac{\sqrt{n_2n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{2,4}-\frac{\sqrt{n_1n_2p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,2}\right\}\left(\boldsymbol{U}_i^{(2)}\phi\big(H(R^*_{2,i})\big)\right)$$

$$+\frac{1}{n_3}\sum_{i=1}^{n_3}\left\{\frac{\sqrt{n_3n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{3,4}-\frac{\sqrt{n_2n_3p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{2,3}-\frac{\sqrt{n_1n_3p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,3}\right\}\left(\boldsymbol{U}_i^{(3)}\phi\big(H(R^*_{3,i})\big)\right)$$

$$+\frac{1}{n_4}\sum_{i=1}^{n_4}\left\{\frac{\sqrt{n_1n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{1,4}-\frac{\sqrt{n_3n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{3,4}-\frac{\sqrt{n_2n_4p}}{\sqrt{NE(\phi^2)}}\boldsymbol{B}'_{2,4}\right\}\left(\boldsymbol{U}_i^{(4)}\phi\big(H(R^*_{4,i})\big)\right),$$

and

$$\boldsymbol{A}=\begin{pmatrix}\sqrt{\lambda_2}I_p & 0_p & 0_p & -\sqrt{\lambda_4}I_p & \sqrt{\lambda_3}I_p & 0_p\\ -\sqrt{\lambda_1}I_p & \sqrt{\lambda_3}I_p & 0_p & 0_p & 0_p & \sqrt{\lambda_4}I_p\\ 0_p & -\sqrt{\lambda_2}I_p & \sqrt{\lambda_4}I_p & 0_p & -\sqrt{\lambda_1}I_p & 0_p\\ 0_p & 0_p & -\sqrt{\lambda_3}I_p & \sqrt{\lambda_1}I_p & 0_p & -\sqrt{\lambda_2}I_p\end{pmatrix}_{4p\times 6p}.$$

Here we can see that there is some pattern in $\boldsymbol{A}_{4p\times 6p}$ such that the pairs of subscripts of $\lambda's$ in columns are (1,2), (2,3), (3,4), (4,1), (1,3), (2,4) which are exactly the same pairs obtained before except for (4,1). We use the pair (4,1) instead of (1,4) because $|1-4| = 3 = 4/2+1$. Also these pairs are arranged in the following order. Suppose we put four numbers in order on a circle. Then the four pairs are the ones from consecutive numbers and (1,3), (2,4) from numbers with absolute difference 2. Note that the first pair shows up in the first column, the second one in the second column etc. and the pair (i,j) takes the i^{th} block of rows and the j^{th} block of rows where the subscripts i and j are placed on diagonal of blocks respectively. All other than these positions contain zero. Also, there is always a negative sign in front of the value related to the first subscript i of (i,j). Using this kind of pattern, we are able to construct the covariance matrix easily without going through all procedures.

Next we prove that $\boldsymbol{V}$ is idempotent to establish the asymptotic null distribution of $\boldsymbol{W}^*_{N,\phi}$. We show this first when p=1 and denote the covariance matrix $\boldsymbol{V}_1 = \boldsymbol{A}_1'\boldsymbol{A}_1$. Let

$$\boldsymbol{A}_1 = \begin{pmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_c' \end{pmatrix}$$

where $\boldsymbol{a}_\alpha'$ is the αth row vector of $\boldsymbol{A}_1$ with dimension $\frac{(c-1)c}{2} \times 1$. Note that

$$\sqrt{\lambda_\alpha}\boldsymbol{a}_\alpha' = -\left(\sqrt{\lambda_1}\boldsymbol{a}_1' + \sqrt{\lambda_2}\boldsymbol{a}_2' + \cdots + \sqrt{\lambda_{\alpha-1}}\boldsymbol{a}_{\alpha-1}' + \sqrt{\lambda_{\alpha+1}}\boldsymbol{a}_{\alpha+1}' + \cdots + \sqrt{\lambda_c}\boldsymbol{a}_c'\right)$$

So that

$$\begin{aligned}
\boldsymbol{V}_1 \cdot \boldsymbol{V}_1 &= (\boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c')(\boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c') \\
&= \boldsymbol{a}_1\boldsymbol{a}_1'\boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_1\boldsymbol{a}_1'\boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_1\boldsymbol{a}_1'\boldsymbol{a}_c\boldsymbol{a}_c' \\
&\quad + \boldsymbol{a}_2\boldsymbol{a}_2'\boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_2\boldsymbol{a}_2'\boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_2\boldsymbol{a}_2'\boldsymbol{a}_c\boldsymbol{a}_c' \\
&\quad + \cdots \\
&\quad + \boldsymbol{a}_c\boldsymbol{a}_c'\boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_c\boldsymbol{a}_c'\boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c'\boldsymbol{a}_c\boldsymbol{a}_c' \\
&= \boldsymbol{a}_1(1-\lambda_1)\boldsymbol{a}_1' + \boldsymbol{a}_1(-\sqrt{\lambda_1\lambda_2})\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_1(-\sqrt{\lambda_1\lambda_c})\boldsymbol{a}_c' \\
&\quad + \boldsymbol{a}_2(-\sqrt{\lambda_1\lambda_2})\boldsymbol{a}_1' + \boldsymbol{a}_2(1-\lambda_2)\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_2(-\sqrt{\lambda_2\lambda_c})\boldsymbol{a}_c' \\
&\quad + \cdots \\
&\quad + \boldsymbol{a}_c(-\sqrt{\lambda_1\lambda_c})\boldsymbol{a}_1' + \boldsymbol{a}_c(-\sqrt{\lambda_2\lambda_c})\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_c(1-\lambda_c)\boldsymbol{a}_c' \\
&= \boldsymbol{a}_1\boldsymbol{a}_1' + \boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c' \\
&\quad - \Big\{\lambda_1\boldsymbol{a}_1\boldsymbol{a}_1' + \lambda_2\boldsymbol{a}_2\boldsymbol{a}_2' + \cdots + \lambda_c\boldsymbol{a}_c\boldsymbol{a}_c' + \sqrt{\lambda_1\lambda_2}(\boldsymbol{a}_1\boldsymbol{a}_2' + \boldsymbol{a}_2\boldsymbol{a}_1') \\
&\quad + \sqrt{\lambda_1\lambda_3}(\boldsymbol{a}_1\boldsymbol{a}_3' + \boldsymbol{a}_3\boldsymbol{a}_1') + \cdots + \sqrt{\lambda_{c-1}\lambda_c}(\boldsymbol{a}_{c-1}\boldsymbol{a}_c' + \boldsymbol{a}_c\boldsymbol{a}_{c-1}')\Big\}
\end{aligned}$$

$$= \boldsymbol{a}_1\boldsymbol{a}_1{}' + \boldsymbol{a}_2\boldsymbol{a}_2{}' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c{}'$$

$$- (\sqrt{\lambda_1}\boldsymbol{a}_1 + \sqrt{\lambda_2}\boldsymbol{a}_2 + \cdots + \sqrt{\lambda_c}\boldsymbol{a}_c)(\sqrt{\lambda_1}\boldsymbol{a}_1{}' + \sqrt{\lambda_2}\boldsymbol{a}_2{}' + \cdots + \sqrt{\lambda_c}\boldsymbol{a}_c{}')$$

$$= \boldsymbol{a}_1\boldsymbol{a}_1{}' + \boldsymbol{a}_2\boldsymbol{a}_2{}' + \cdots + \boldsymbol{a}_c\boldsymbol{a}_c{}'$$

$$= \boldsymbol{A}_1{}' \cdot \boldsymbol{A}_1$$

$$= \boldsymbol{V}_1$$

Since $\boldsymbol{V} = \boldsymbol{V}_1 \otimes \boldsymbol{I}_p$, and

$$\boldsymbol{V} \cdot \boldsymbol{V} = (\boldsymbol{V}_1 \otimes \boldsymbol{I}_p)(\boldsymbol{V}_1 \otimes \boldsymbol{I}_p) = (\boldsymbol{V}_1 \cdot \boldsymbol{V}_1) \otimes (\boldsymbol{I}_p \cdot \boldsymbol{I}_p) = \boldsymbol{V}_1 \otimes \boldsymbol{I}_p = \boldsymbol{V},$$

$\boldsymbol{V}$ is idempotent. Therefore the asymptotic null distribution of $\boldsymbol{W}^*_{N,\phi}$ is chi-squared distribution with degrees of freedom tr$(\boldsymbol{V})$ (see Searle 1971), where

$$tr(\boldsymbol{V}) = (\lambda_1 + \lambda_2)p + (\lambda_2 + \lambda_3)p + \cdots + (\lambda_{c-1} + \lambda_c)p + \cdots +$$

$$= (c-1)p(\lambda_1 + \lambda_2 + \cdots + \lambda_c)$$

$$= (c-1)p$$

This completes the proof.

5.3 Asymptotic Distribution of $\boldsymbol{W}_{N,\phi}$ under Contiguous Alternatives

In finding the asymptotic distribution of $\boldsymbol{W}_{N,\phi}$ under the sequence of alternatives approaching the null hypothesis, we consider the exponential power family in section 3.2 as one of the classes of elliptically symmetric distributions. We assume $\boldsymbol{X}_1^{(\alpha)},\ldots,\boldsymbol{X}_{n_\alpha}^{(\alpha)}$ are iid from f$(\boldsymbol{x}^{(\alpha)} - \frac{\boldsymbol{d}_\alpha}{\sqrt{N}})$ where $\boldsymbol{d}_\alpha = (d_{\alpha 1}, \ldots, d_{\alpha p})'$ satisfies $\sum_{\alpha=1}^{c} \lambda_\alpha \boldsymbol{d}_\alpha = \boldsymbol{0}$ and where f is given in (2.2). Then the rationale of Hajek and Sidak (1967, pp. 208-213) shows that the alternatives are contiguous to the null hypothesis. Under the null hypothesis, the approximation to the log-likelihood function is equal

to

$$\boldsymbol{T}_N^* = \frac{1}{\sqrt{N}}\left\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha} \boldsymbol{d}_\alpha{}' \frac{f'(\boldsymbol{X}_i^{(\alpha)})}{f(\boldsymbol{X}_i^{(\alpha)})}\right\}$$

$$= \frac{2\nu}{c^\nu\sqrt{N}}\left\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha} \boldsymbol{d}_\alpha{}' \boldsymbol{X}_i^{(\alpha)} (\boldsymbol{X}_i^{(\alpha)\prime}\boldsymbol{X}_i^{(\alpha)})^{\nu-1}\right\}$$

$$= \frac{2\nu}{c^\nu\sqrt{N}}\left\{\sum_{\alpha=1}^{c}\sum_{i=1}^{n_\alpha} \boldsymbol{d}_\alpha{}' (R_{\alpha,i}^*)^{\nu-1/2}\boldsymbol{U}_i^{(\alpha)}\right\}$$

Now we establish the following theorem.

Theorem 5.3.1 Under the sequences of alternatives,

$$\boldsymbol{W}_{N,\phi} \xrightarrow{d} \chi^2_{(c-1)p}(\delta),$$

where

$$\delta = \frac{4\nu^2}{pc^{2\nu}E(\phi^2)} E^2\left(\phi\big(H(R_{1,i}^*)\big)(R_{1,i}^*)^{\nu-1/2}\right)\sum_{k=1}^{c}\lambda_k \boldsymbol{d}_k{}'\boldsymbol{d}_k$$

Proof of Theorem 5.3.1 Let $\boldsymbol{a} = (a_1, a_2)'$ be an arbitrary fixed vector of constants none of which are zero. We first find the joint limiting distribution of $\boldsymbol{T}_N^*$ and $\boldsymbol{S}_N^* =$

$\boldsymbol{B}'\boldsymbol{S}_N$.

$$
\begin{aligned}
\boldsymbol{a}'\begin{pmatrix}\boldsymbol{S}_N^*\\ \boldsymbol{T}_N^*\end{pmatrix} &= a_1\boldsymbol{S}_N^* + a_2\boldsymbol{T}_n^*\\
&= \sum_{i=1}^{n_1}\Bigg[\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{1,i}^*)^{\nu-1/2}\boldsymbol{d}_1 + a_1\phi\Big(H(R_{1,i}^*)\Big)\\
&\qquad\times\Big(\frac{\sqrt{n_2p}}{\sqrt{n_1NE(\phi^2)}}\boldsymbol{B}_{1,2} + \cdots - \frac{\sqrt{n_cp}}{\sqrt{n_1NE(\phi^2)}}\boldsymbol{B}_{1,c}\Big)\Bigg]'\boldsymbol{U}_i^{(1)}\\
&\qquad+\sum_{i=1}^{n_2}\Bigg[\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{2,i}^*)^{\nu-1/2}\boldsymbol{d}_2 + a_1\phi\Big(H(R_{2,i}^*)\Big)\\
&\qquad\times\Big(\frac{\sqrt{n_3p}}{\sqrt{n_2NE(\phi^2)}}\boldsymbol{B}_{2,3} + \cdots - \frac{\sqrt{n_1p}}{\sqrt{n_2NE(\phi^2)}}\boldsymbol{B}_{1,2}\Big)\Bigg]'\boldsymbol{U}_i^{(2)}\\
&\qquad+\cdots\\
&\qquad+\sum_{i=1}^{n_c}\Bigg[\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{c,i}^*)^{\nu-1/2}\boldsymbol{d}_c + a_1\phi\Big(H(R_{c,i}^*)\Big)\\
&\qquad\times\Big(\frac{\sqrt{n_1p}}{\sqrt{n_cNE(\phi^2)}}\boldsymbol{B}_{1,c} + \cdots - \frac{\sqrt{n_{c-1}p}}{\sqrt{n_cNE(\phi^2)}}\boldsymbol{B}_{c-1,c}\Big)\Bigg]'\boldsymbol{U}_i^{(c)}
\end{aligned}
$$

Since $\mathrm{E}_{H_0}[S_N^*]$ and $\mathrm{E}_{H_0}[T_N^*]$ are zero, $\mathrm{E}_{H_0}[\boldsymbol{a}'(S_N^*, T_N^*)'] = 0$.

$$
\begin{aligned}
&\mathrm{V}_{H_0}\left[\boldsymbol{a}'(S_N^*, T_N^*)'\right] \\
&\quad = a_1^2 \mathrm{V}_{H_0}\left[S_N^*\right] + a_2^2 \mathrm{V}_{H_0}\left[T_N^*\right] + 2a_1a_2 \mathrm{Cov}_{H_0}\left[S_N^*, T_N^*\right] \\
&\quad = \mathrm{E}_{H_0}\left[(\boldsymbol{a}'(S_N^*, T_N^*)')^2\right] \\
&\quad = n_1 \sum_{t=1}^{p} E_{H_o}\Bigg[\Bigg\{\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{1,i}^*)^{\nu-1/2} d_{1t} + a_1\phi\Big(H(R_{1,i}^*)\Big) \\
&\qquad \times \Big(\frac{\sqrt{n_2 p}}{\sqrt{n_1 N E(\phi^2)}} b_t^{1,2} + \cdots - \frac{\sqrt{n_c p}}{\sqrt{n_1 N E(\phi^2)}} b_t^{1,c}\Big)\Bigg\}^2\Bigg] E_{H_o}(U_{1t}^{(1)})^2 \\
&\qquad + n_2 \sum_{t=1}^{p} E_{H_o}\Bigg[\Bigg\{\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{2,i}^*)^{\nu-1/2} d_{2t} + a_1\phi\Big(H(R_{2,i}^*)\Big) \\
&\qquad \times \Big(\frac{\sqrt{n_2 p}}{\sqrt{n_2 N E(\phi^2)}} b_t^{2,3} + \cdots - \frac{\sqrt{n_c p}}{\sqrt{n_2 N E(\phi^2)}} b_t^{1,2}\Big)\Bigg\}^2\Bigg] E_{H_o}(U_{2t}^{(2)})^2 \\
&\qquad + \cdots \\
&\qquad + n_c \sum_{t=1}^{p} E_{H_o}\Bigg[\Bigg\{\frac{2a_2\nu}{c^\nu\sqrt{N}}(R_{c,i}^*)^{\nu-1/2} d_{ct} + a_1\phi\Big(H(R_{c,i}^*)\Big) \\
&\qquad \times \Big(\frac{\sqrt{n_2 p}}{\sqrt{n_c N E(\phi^2)}} b_t^{1,c} + \cdots - \frac{\sqrt{n_c p}}{\sqrt{n_c N E(\phi^2)}} b_t^{c-1,c}\Big)\Bigg\}^2\Bigg] E_{H_o}(U_{ct}^{(c)})^2,
\end{aligned}
$$

where $b_t^{\alpha,\beta}$ is the t^{th} element of the matrix $B_{\alpha,\beta}$. So that, using $E_{H_o}(U_{\alpha t}^2) = \frac{1}{p}$ gives

$$\sigma_1^2 = \frac{1}{N}\Big[\left(\sqrt{n_2}\boldsymbol{B}'_{1,2} + \cdots - \sqrt{n_c}\boldsymbol{B}'_{1,c}\right)'\left(\sqrt{n_2}\boldsymbol{B}'_{1,2} + \cdots - \sqrt{n_c}\boldsymbol{B}'_{1,c}\right)$$

$$+ \left(\sqrt{n_3}\boldsymbol{B}'_{2,3} + \cdots - \sqrt{n_1}\boldsymbol{B}'_{1,2}\right)'\left(\sqrt{n_3}\boldsymbol{B}'_{2,3} + \cdots - \sqrt{n_1}\boldsymbol{B}'_{1,2}\right)$$

$$+ \cdots +$$

$$+ \left(\sqrt{n_1}\boldsymbol{B}'_{1,c} + \cdots - \sqrt{n_{c-1}}\boldsymbol{B}'_{c-1,c}\right)'\left(\sqrt{n_1}\boldsymbol{B}'_{1,c} + \cdots - \sqrt{n_{c-1}}\boldsymbol{B}'_{c-1,c}\right)\Big]$$

$$= \left(\begin{pmatrix} \sqrt{n_2/N}I_p & \cdots & \cdots & -\sqrt{n_c/N}I_p \\ -\sqrt{n_1/N}I_p & \sqrt{n_3/N}I_p & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \sqrt{n_1/N}I_p \end{pmatrix}\boldsymbol{B}\right)'$$

$$\cdot \left(\begin{pmatrix} \sqrt{n_2/N}I_p & \cdots & \cdots & -\sqrt{n_c/N}I_p \\ -\sqrt{n_1/N}I_p & \sqrt{n_3/N}I_p & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \sqrt{n_1/N}I_p \end{pmatrix}\boldsymbol{B}\right),$$

and hence we see that $\sigma_1^2 \to \boldsymbol{B}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}'\boldsymbol{V}\boldsymbol{B}$ as $N \to \infty$.

$$\sigma_{12} = \frac{2\nu}{c^\nu N\sqrt{pE(\phi^2)}} E_{H_o}\Big[\phi\big(H(R^*_{1,i_1})\big)(R^*_{1,i_1})^{\nu-1/2}\Big] \times$$

$$\Big[n_1\Big(\frac{\sqrt{n_2}}{\sqrt{n_1}}\boldsymbol{B}'_{1,2}\boldsymbol{d}_1 + \cdots - \frac{\sqrt{n_c}}{\sqrt{n_1}}\boldsymbol{B}'_{1,c}\boldsymbol{d}_1\Big)$$

$$+ n_2\Big(\frac{\sqrt{n_3}}{\sqrt{n_2}}\boldsymbol{B}'_{2,3}\boldsymbol{d}_2 + \cdots - \frac{\sqrt{n_1}}{\sqrt{n_2}}\boldsymbol{B}'_{1,2}\boldsymbol{d}_2\Big)$$

$$+ \cdots$$

$$+ n_c\Big(\frac{\sqrt{n_1}}{\sqrt{n_c}}\boldsymbol{B}'_{1,c}\boldsymbol{d}_c + \cdots - \frac{\sqrt{n_{c-1}}}{\sqrt{n_c}}\boldsymbol{B}'_{c-1,c}\boldsymbol{d}_c\Big)\Big]$$

$$= \frac{2\nu}{c^\nu N\sqrt{pE(\phi^2)}} E_{H_o}\Big[\phi\big(H(R^*_{1,i_1})\big)(R^*_{1,i_1})^{\nu-1/2}\Big](\boldsymbol{B}'_{1,2}, \boldsymbol{B}'_{2,3}, \ldots, \boldsymbol{B}'_{c-1,c}) \cdot$$

$$\begin{pmatrix} -\sqrt{n_1 n_2} I_p & \sqrt{n_1 n_2} I_p & \boldsymbol{0}_p & \cdots & \boldsymbol{0}_p \\ \boldsymbol{0}_p & -\sqrt{n_2 n_3} I_p & -\sqrt{n_2 n_3} I_p & \boldsymbol{0}_p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{n_c n_1} I_p & \vdots & \vdots & \vdots & -\sqrt{n_c n_1} I_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_c \end{pmatrix}.$$

Also

$$\sigma_2^2 = \frac{4\nu^2}{c^{2\nu} pN} E_{H_o}\Big((R^*_{1,i_1})^{2\nu-1}\Big)\Big(n_1\boldsymbol{d}_1'\boldsymbol{d}_1 + n_2\boldsymbol{d}_2'\boldsymbol{d}_2 + \cdots + n_c\boldsymbol{d}_c'\boldsymbol{d}_c\Big).$$

Then by Lecam's third Lemma, we have, under the sequence of alternatives,

$$\boldsymbol{S}^*_N \xrightarrow{d} N\big(\mu^*_a, \boldsymbol{B}'\boldsymbol{V}\boldsymbol{B}\big)$$

where

$$\mu^*_a = \frac{2\nu}{c^\nu \sqrt{pE(\phi^2)}} E_{H_o}\Big[\phi\big(H(R^*_{1,i})\big)(R^*_{1,i})^{\nu-1/2}\Big]\boldsymbol{B}'\boldsymbol{G}\boldsymbol{d},$$

and

$$
G = \begin{pmatrix}
-\sqrt{n_1 n_2} I_p & \sqrt{n_1 n_2} I_p & 0_p & \cdots & 0_p \\
0_p & -\sqrt{n_2 n_3} I_p & -\sqrt{n_2 n_3} I_p & 0_p & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\sqrt{n_c n_1} I_p & \vdots & \vdots & \vdots & -\sqrt{n_c n_1} I_p \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}_{\frac{(c-1)cp}{2} \times cp} .
$$

For example, when $c = 4$, G is of the following form,

$$
G = \begin{pmatrix}
-\sqrt{\lambda_1 \lambda_2} I_p & \sqrt{\lambda_1 \lambda_2} I_p & 0_p & 0_p \\
0_p & -\sqrt{\lambda_2 \lambda_3} I_p & -\sqrt{\lambda_2 \lambda_3} I_p & 0_p \\
0_p & 0_p & -\sqrt{\lambda_3 \lambda_4} & \sqrt{\lambda_3 \lambda_4} \\
\sqrt{\lambda_4 \lambda_1} & 0_p & 0_p & -\sqrt{\lambda_4 \lambda_1} \\
-\sqrt{\lambda_1 \lambda_3} & 0_p & \sqrt{\lambda_1 \lambda_3} & 0_p \\
0_p & -\sqrt{\lambda_2 \lambda_4} & 0_p & \sqrt{\lambda_2 \lambda_4}
\end{pmatrix}_{6p \times 4p} .
$$

So that under the contiguous alternatives

$$
S_N \xrightarrow{d} N\left(\frac{2\nu}{c^{\nu} \sqrt{pE(\phi^2)}} E_{H_o} \left[\phi\Big(H(R^*_{1,i})\Big) (R^*_{1,i})^{\nu - 1/2} \right] Gd, V \right).
$$

This gives

$$
W^*_{N,\phi} \xrightarrow{d} \chi^2_{(c-1)p}\left(\left(\frac{4\nu^2}{c^{2\nu} pE(\phi^2)} E^2_{H_o} \left[\phi\Big(H(R^*_{1,i})\Big) (R^*_{1,i})^{\nu - 1/2} \right] d' G' G d \right) .
$$

To simplify the form of noncentrality parameter of the chi-square distribution, we use the assumption that $\sum_{k=1}^{c} \lambda_k \boldsymbol{d}_k = \boldsymbol{0}$. Since

$$\begin{aligned}
\boldsymbol{d}'\boldsymbol{G}'\boldsymbol{G}\boldsymbol{d} &= \boldsymbol{d}' \begin{pmatrix} \lambda_1(1-\lambda_1)I_p & -\lambda_1\lambda_2 I_p & -\lambda_1\lambda_3 & \cdots & -\lambda_1\lambda_c \\ -\lambda_1\lambda_2 I_p & \lambda_2(1-\lambda_2)I_p & -\lambda_2\lambda_3 & \cdots & -\lambda_2\lambda_c \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda_c\lambda_1 & -\lambda_c\lambda_1 & -\lambda_c\lambda_3 & \cdots & \lambda_c(1-\lambda_c) \end{pmatrix} \boldsymbol{d} \\
&= \sum_{k=1}^{c} \lambda_k \boldsymbol{d}'\boldsymbol{d} - \Big[\Big(\sum_{k=1}^{c} \lambda_k \boldsymbol{d}\Big)'\Big(\sum_{k=1}^{c} \lambda_k \boldsymbol{d}\Big)\Big] \\
&= \sum_{k=1}^{c} \lambda_k \boldsymbol{d}'\boldsymbol{d},
\end{aligned}$$

we have the desired result.

5.4 Pitman Asymptotic Relative Efficiency

We use Pitman asymptotic relative efficiency to make a comparison between $\boldsymbol{W}_{N,\phi}$ and the generalized Hotelling's T^2. Since these statistics are affine-invariant, we may, without loss of generality, make the assumption that the common covariance matrix is the identity. Hence we will apply the asymptotic result under the contiguous alternatives discussed in the previous section. Note that the asymptotic distribution of the generalized Hotelling's T^2 under the same sequence of contiguous alternatives is Chi-squared with noncentrality parameter $\sum_{\alpha=1}^{c} \lambda_\alpha \boldsymbol{d}'\boldsymbol{d}$ (Puri and Sen, 1970 pp.212). Thus

$$\mathrm{ARE}(\boldsymbol{W}_{N,\phi}, T^2) = \frac{4\nu^2}{pc^{2\nu}E(\phi^2)} E^2\Big(\phi\big(H(R_{1,i}^*)\big)(R_{1,i}^*)^{\nu-1/2}\Big). \tag{5.3}$$

When the score functions $\phi_1(u) = 1$ and $\phi_2(\mathrm{u}) = \mathrm{u}$, the asymptotic relative efficiencies are identical to the one-sample asymptotic relative efficiencies reported in Randles (1989) and in Peters and Randles(1990) respectively. So that after evaluating the

expection we have

$$\mathrm{ARE}(\boldsymbol{W}_{N,\phi_1}, T^2) = \frac{4\nu^2\Gamma^2\left(\frac{2\nu+p-1}{2\nu}\right)\Gamma\left(\frac{p+2}{2\nu}\right)}{p^2\Gamma^3\left(\frac{p}{2\nu}\right)}, \tag{5.4}$$

and

$$\mathrm{ARE}(\boldsymbol{W}_{N,\phi_2}, T^2) = \frac{12\nu^2\Gamma\left(\frac{p+2}{2\nu}\right)\Gamma^2\left(\frac{2p+2\nu-1}{2\nu}\right)\beta^2\left(\frac{1}{2}; \frac{p}{2\nu}; \frac{2p+2\nu-1}{2\nu}\right)}{p^2\Gamma^5\left(\frac{p}{2\nu}\right)}, \tag{5.5}$$

where $\Gamma(\mathrm{z}) = \int_0^\infty x^{z-1}e^{-x}dx, z > 0$, and $\beta(\frac{1}{2}; a; b) = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx, a, b > 0$. It is of interest to note that $\lim_{\nu\to\infty} \mathrm{ARE}(\boldsymbol{W}_{N,\phi_1}, T^2) = \frac{p}{p+2}$ (≤ 1) and $\lim_{\nu\to\infty}$ $\mathrm{ARE}(\boldsymbol{W}_{N,\phi_2}, T^2) = \frac{3p}{p+2}$ (≥ 1) for fixed p. Thus we expect that for light-tailed distributions $\boldsymbol{W}_{N,\phi_2}$ performs better than both $\boldsymbol{W}_{N,\phi_1}$ and the genelalized Hotelling's T^2 when p > 1. In Table we present the Pitman asymptotic relative efficiencies of $\boldsymbol{W}_{N,\phi}$ relative to the genelalized Hotelling's T^2 for score functions and selected values of ν and p.

Table 5.1. $\mathrm{ARE}(\boldsymbol{W}_{N,\phi_1}, T^2)$ and $\mathrm{ARE}(\boldsymbol{W}_{N,\phi_2}, T^2)$

	ν									
	0.1		0.5		1.0		2.0		5.0	
p	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	ϕ_2
1	252252	739	2.00	1.50	.637	.954	.411	.873	.347	.907
2	367	25	1.50	1.13	.785	.985	.590	1.051	.519	1.221
3	54.8	7.40	1.33	1.00	.849	.975	.688	1.099	.620	1.346
4	21.0	3.93	1.25	0.94	.884	.961	.749	1.109	.687	1.395
5	11.7	2.67	1.20	0.90	.905	.949	.790	1.106	.734	1.412

As can be seen $\boldsymbol{W}_{N,\phi_2}$ is more efficient than $\boldsymbol{W}_{N,\phi_1}$ and the generalized Hotelling's T^2 for light-tailed distributions ($\nu = 2$ and 5) and p> 1. Even under the multivariate

normal distribution and the distributions close to the normal $\boldsymbol{W}_{N,\phi_2}$ continues to be efficient relative to the generalized Hotelling's T^2. For heavy-tailed distributions (ν = 0.1 and 0.5), $\boldsymbol{W}_{N,\phi_1}$ beats both $\boldsymbol{W}_{N,\phi_2}$ and the generalized Hotelling's T^2.

CHAPTER 6
LOCATION PROBLEM : MONTE CARLO STUDY AND EXAMPLE

In this chapter we display results from a Monte Carlo study when the number of samples is $c = 3$, the dimension is p = 3, and each sample size is $n_\alpha = 15$, $\alpha = 1, 2, 3$. The five statistics W_{N,ϕ_1}, W_{N,ϕ_2}, Lawley-Hotelling's generalized T^2, and the statistics L_N and H (described in chapter 1) were compared using samples from the exponential power family and the multivariate t-distributions family. Specifically, the distributions considered are the multivariate normal ($\nu = 1$) with correlation of 0.9, the multivariate Cauchy ($df = 1$) and the exponential power family with $\nu = 25$. The number of repetitions at each case was 1000. In each Monte Carlo simulation, the proportion of times out of 1000 in which each test statistic exceeded its asymptotic critical value is reported. The asymptotic null distribution χ^2_6 is used to determine the critical value for tests W_{N,ϕ_1}, W_{N,ϕ_2}, L_N and H. For Lawley-Hotelling's generalized T^2, the test rejects H_0 if T^2 exceeds $\frac{84}{13}F_{6,78}$ where F_{ν_1,ν_2} denotes the upper 5 percentile of an F distribution with ν_1 and ν_2 degrees of freedom. All simulation programs were written in the C programming language and the same routines as described in chapter 4 were used on a DECstation.

In Table 6.1 we present the results of the Monte Carlo study for various values of the locations $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$. The first sample was selected in each case from the indicated population centered at the origin. The second and third samples were selected from the same population but centered at shifted values $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ respectively as indicated in Table 6.1. The locations for the second and third population were selected so that the results would represent a reasonable range of powers under the

alternative hypotheses. For the multivariate Cauchy ($df = 1$) W_{N,ϕ_1} performs better than W_{N,ϕ_2}, L_N and H. For the light-tailed distribution ($\nu = 25$), W_{N,ϕ_2} has the greatest powers. When $\nu = 1$, W_{N,ϕ_1} and W_{N,ϕ_2} perform quite well with respect to T^2. The L_N seems to perform well for the heavy-tailed distribution but its performance can become poor because of the lack of affine-invariance. The statistic H has smaller powers than W_{N,ϕ_1} and W_{N,ϕ_2} for $\nu = 1$ and $\nu = 25$ although it beats W_{N,ϕ_2} for $df = 1$. In general, W_{N,ϕ_1} appears to perform better than H.

We make another comparison among those five statistics using the same example Hettmansperger and Oja (1994) took from Seber (1984). The original data consist of observations on men in four weight categories. A portion of data (N = 36 observations) was taken in three weight categories ($n_1 = 10$, $n_2 = 12$ and $n_3 = 14$). Urine samples were taken from each man. The variables to consider are amount of phosphate (mg/ml) and amount of calcium (mg/ml). The data and the result of analysis are given in Table 6.2 and 6.3 respectively. Here the approximate distribution of Lawley Hotelling's generalized T^2 is $F_{4,62}$ and the others have an approximate χ^2_4 distribution. We see that the statistics W_{N,ϕ_1} and W_{N,ϕ_2} perform very well with corresponding P-values 0.0076 and 0.0046.

Table 6.1. Monte Carlo Results with n = 15, reps = 1000

			Statistics				
$\boldsymbol{\theta}_1$	$\boldsymbol{\theta}_2$	$\boldsymbol{\theta}_3$	W_{N,ϕ_1}	W_{N,ϕ_2}	T^2	L_N	H
					$df = 1$		
(0,0,0)	(0,0,0)	(0,0,0)	0.0580	0.0590	0.0120	0.0450	0.0380
(0,0,0)	(0.4,0.4,0.4)	(0,-0.4,0)	0.2640	0.1990	0.0350	0.1990	0.2130
(0,0,0)	(0.6,0.6,0.6)	(0,-0.6,0)	0.4630	0.2990	0.0750	0.3490	0.4170
(0,0,0)	(0.8,0.8,0.8)	(0,-0.8,0)	0.6880	0.4050	0.0970	0.5660	0.6600
					$\nu = 1$, correlations = 0.9		
(0,0,0)	(0,0,0)	(0,0,0)	0.0620	0.0630	0.0490	0.0300	0.0370
(0,0,0)	(-0.1,-0.1,0)	(0.1,0.1,0)	0.1390	0.1520	0.1410	0.0540	0.1050
(0,0,0)	(-0.2,-0.2,0)	(0.2,0.2,0)	0.4300	0.4590	0.5050	0.1000	0.3860
(0,0,0)	(-0.3,-0.3,0)	(0.3,0.3,0)	0.8080	0.8400	0.8970	0.2030	0.7800
					$\nu = 25$		
(0,0,0)	(0,0,0)	(0,0,0)	0.0650	0.0710	0.0480	0.0420	0.0380
(0,0,0)	(0.2,0.2,0.2)	(0,-0.2,0)	0.1620	0.1740	0.1500	0.1230	0.1260
(0,0,0)	(0.4,0.4,0.4)	(0,-0.4,0)	0.2070	0.3670	0.2780	0.1300	0.1720
(0,0,0)	(0.6,0.6,0.6)	(0,-0.6,0)	0.3960	0.7300	0.6200	0.2610	0.3520

Table 6.2. Urine Data in men in three weight groups

Group	Phosphate (mg/ml)	Calcium (mg/ml)	Group	Phosphate (mg/ml)	Calcium (mg/ml)
1	2.05	0.222	2	2.30	0.275
1	1.05	0.267	2	2.35	0.210
1	5.90	0.093	2	2.35	0.050
1	4.25	0.147	2	2.50	0.143
1	3.85	0.217	3	1.50	0.153
1	2.45	0.418	3	1.65	0.203
1	1.70	0.323	3	1.40	0.074
1	1.80	0.205	3	1.65	0.155
1	3.65	0.348	3	0.90	0.155
1	2.25	0.320	3	1.60	0.129
2	1.50	0.104	3	2.45	0.245
2	1.65	0.245	3	1.65	0.422
2	0.90	0.097	3	1.65	0.063
2	1.75	0.174	3	1.25	0.042
2	1.40	0.210	3	1.05	0.030
2	1.20	0.275	3	2.70	0.194
2	1.90	0.170	3	1.60	0.139
2	1.65	0.164	3	0.85	0.046

Table 6.3. Statistical Analysis of Urine Data

	Statistics				
	W_{N,ϕ_1}	W_{N,ϕ_2}	T^2	L_N	H
Value	13.8931	15.0518	6.262	12.2479	12.511
P-value	0.0076	0.0046	0.0003	0.0156	0.0139

APPENDIX
SOURCE CODE

Some of the Monte Carlo programs are listed in this appendix. Included are yong.c for the independence problem and signrank.c and location.c for the location problem. Other subroutines such as interdir.c, and quadrant.c are used and can be found in Geiser (1993).

.0.1 yong.c

```
#include <stdio.h>
#include <math.h>
#include "ranlib.h"
#include "general.h"

#define NOMINAL 0.05

/*
 * This is the main routine which does the repetitions
 * and counts the number of times each of the competing
 * statistics falls in the rejection region.
 */

char *usage = "Usage: %s [p or n] dim N reps delta nu seedfile\n";
char *cmd;

main(int argc, char **argv)
{
  double delta, nu;
  int i, ii, j, jj, dim, r1, r2, r3, r12, r23, r31, r, N, reps;
  double k, k1, k2, k3, t, num1, num2, num3, num4;
  double Q1, Q2, PS, PSa, L1, L2, L3, L12, L, La;
  double X;
  VEC *ER1, *ER2, *ER3, *ER12,*EI1, *EI2, *EI3, *EI12;
  MAT *A1, *A2, *A3, *B1, *B2, *B3, *B12, *B23, *B31,
```

```
    *TB12, *TB23, *TB31, *S11, *S22, *S33, *S12, *S21, *S31, *S13,
    *S23, *T1, *T2, *T3, *T4, *TMP1, *TMP2, *TMP3, *TMP4, *TMP5,
    *TMP6, *TMP7, *TMP8, *SA, *SB, *SC, *SAB;
MAT *TEMP1, *TEMP2, *TEMP3, *TEMP4, *TEMP5, *TEMP6, *TEMP7, *TEMP8,
    *TEMP9, *TEMP10, *TEMP11, *TEMP12, *TEMP13, *TEMP14, *TEMP15,
    *M1, *M2, *M3, *M4, *M5, *M6, *IDA, *IDB, *IDC, *IDAB,
    *OUTA, *OUTB, *OUTC, *OUTAB;
VEC *THETA1, *THETA2, *THETA3, *TEMPa, *TEMPb, *TEMPc, *TEMPd,
    *TEMPe, *TEMPf, *TEMPg;
int cL, cLa, cPS, cPSa, cQ1, cQ2;
int iL, iLa, iPS, iPSa, iQ1, iQ2;

char option;

long seed1, seed2, g;
char seedfile[80];
FILE *seeds;

cmd = *argv;

if (argc < 8) bye();

if (sscanf(argv[1],"%s",&option) != 1) bye();

if (sscanf(argv[2],"%u",&dim) != 1) bye();
if (dim < 1 || dim > MAX_DIM) {
  printf("Invalid value of dim\n");
  exit(0);
}

if (sscanf(argv[3],"%u",&N) != 1) bye();
if (N < 2 * dim + 1 || N > MAX_N) {
  printf("Invalid value of N\n");
  exit(0);
}

if (sscanf(argv[4],"%u",&reps) != 1) bye();
if (reps < 1) {
  printf("Invalid value of reps\n");
  exit(0);
}

if (sscanf(argv[5],"%lf",&delta) != 1) bye();
if (delta < 0.0 || delta > 0.5) {
  printf("Invalid value of delta\n");
```

```
  exit(0);
}

if (sscanf(argv[6],"%lf",&nu) ≠ 1) bye();
if (nu ≤ 0.0) {
  printf("Invalid value of nu\n");
  exit(0);
}

if (sscanf(argv[7],"%s",&seedfile) ≠ 1) bye();

/*
 * Set generator number and current seeds
 */

seeds = fopen(seedfile,"rt");
while(fscanf(seeds,"%ld %ld %ld",&g,&seed1,&seed2) ≠ EOF);
fclose(seeds);
gscgn(1,&g);
setall(seed1,seed2);

r1 = r2 = dim;
r3 = 3;
r = r1 + r2 + r3;
r12 = r1 + r2;
r23 = r2 + r3;
r31 = r3 + r1;
A1 = get_mat(N,r1);
A2 = get_mat(N,r2);
A3 = get_mat(N,r3);
B1 = get_mat(N,r1);
B2 = get_mat(N,r2);
B3 = get_mat(N,r3);
B12 = get_mat(N,r12);
B23 = get_mat(N,r23);
B31 = get_mat(N,r31);
TB12 = get_mat(r12,r12);
TB23 = get_mat(r23,r23);
TB31 = get_mat(r31,r31);
S11 = get_mat(r1,r1);
S22 = get_mat(r2,r2);
S33 = get_mat(r3,r3);
S12 = get_mat(r1,r2);
S21 = get_mat(r2,r1);
S13 = get_mat(r1,r3);
```

```
S31 = get_mat(r3,r1);
S23 = get_mat(r2,r3);

T1 = get_mat(r1,r12);
T2 = get_mat(r2,r12);
T3 = get_mat(r12,r12);
T4 = get_mat(r12,r3);
TMP1 = get_mat(r1,r2);
TMP2 = get_mat(r2,r1);
TMP3 = get_mat(r2,r3);
TMP4 = get_mat(r3,r2);
TMP5 = get_mat(r3,r1);
TMP6 = get_mat(r1,r3);
TMP7 = get_mat(r12,r3);
TMP8 = get_mat(r3,r12);
SA = get_mat(r1,r1);
SB = get_mat(r2,r2);
SC = get_mat(r3,r3);
SAB = get_mat(r12,r12);
IDA = get_mat(r1,r1);
IDB = get_mat(r2,r2);
IDC = get_mat(r3,r3);
IDAB = get_mat(r12,r12);
OUTA = get_mat(r1,r1);
OUTB = get_mat(r2,r2);
OUTC = get_mat(r3,r3);
OUTAB = get_mat(r12,r12);

M1 = get_mat(r2,r1);
M2 = get_mat(r3,r1);
M3 = get_mat(r1,r2);
M4 = get_mat(r3,r2);
M5 = get_mat(r1,r3);
M6 = get_mat(r2,r3);

TEMP1 = get_mat(N,r2);
TEMP2 = get_mat(N,r1);
TEMP3 = get_mat(N,r3);
TEMP4 = get_mat(N,r1);
TEMP5 = get_mat(N,r1);

TEMP6 = get_mat(N,r1);
TEMP7 = get_mat(N,r2);
TEMP8 = get_mat(N,r3);
TEMP9 = get_mat(N,r2);
```

```
TEMP10 = get_mat(N,r2);

TEMP11 = get_mat(N,r1);
TEMP12 = get_mat(N,r3);
TEMP13 = get_mat(N,r2);
TEMP14 = get_mat(N,r3);
TEMP15 = get_mat(N,r3);

THETA1 = get_vec(r1);
THETA2 = get_vec(r2);
THETA3 = get_vec(r3);

ER1 = get_vec(r1);
ER2 = get_vec(r2);
ER3 = get_vec(r3);
ER12 = get_vec(r12);
EI1 = get_vec(r1);
EI2 = get_vec(r2);
EI3 = get_vec(r3);
EI12 = get_vec(r12);
TEMPa = get_vec(N);
TEMPb = get_vec(N);
TEMPc = get_vec(N);
TEMPd = get_vec(r1);
TEMPe = get_vec(r2);
TEMPf = get_vec(r12);
TEMPg = get_vec(r3);

  ones_mat(M1);
  ones_mat(M2);
  ones_mat(M3);
  ones_mat(M4);
  ones_mat(M5);
  ones_mat(M6);

/* Constants needed for statistics */

t=(r*r*r - (r1*r1*r1 + r2*r2*r2 +r3*r3*r3))/
  (3*(r*r - r1*r1 - r2*r2 - r3*r3));
k = N - 3/2 - t;
k1 = N - (r1 + r2 + 3)/2.0;
k2 = N - (r2 + r3 + 3)/2.0;
k3 = N - (r3 + r1 + 3)/2.0;

X = critchi(NOMINAL,r1*r2 + r2*r3 + r3*r1);
```

```
cL = cLa = cQ1 = cQ2 = cPS = cPSa = 0;


 for (ii = 0; ii < r1; ++ii)
      for (jj = 0; jj < r1; ++jj)
      {
       IDA->me[ii][jj] = 0.0;
       if(ii == jj) IDA->me[ii][jj] = 1.0;
      }

  for (ii = 0; ii < r2; ++ii)
      for (jj = 0; jj < r2; ++jj)
      {
       IDB->me[ii][jj] = 0.0;
       if(ii == jj) IDB->me[ii][jj] = 1.0;
      }

  for (ii = 0; ii < r3; ++ii)
      for (jj = 0; jj < r3; ++jj)
      {
       IDC->me[ii][jj] = 0.0;
       if(ii == jj) IDC->me[ii][jj] = 1.0;
      }
  for (ii = 0; ii < r12; ++ii)
      for (jj = 0; jj < r12; ++jj)
      {
       IDAB->me[ii][jj] = 0.0;
       if(ii == jj) IDAB->me[ii][jj] = 1.0;
      }


if (option == 'p') {
printf("\n\n       L      La      PS     PSa     Q1     Q2\n");
}

for(j = reps; j > 0; --j) {

  /*
   * Generate the initial random vectors
   */

  rand_dat(A1,nu);
```

```
rand_dat(A2,nu);
rand_dat(A3,nu);

/*
 * Create dependence via model
 */

sm_mlt(delta,A2,TEMP1);
m_mlt(TEMP1,M1,TEMP2);
sm_mlt(delta,A3,TEMP3);
m_mlt(TEMP3,M2,TEMP4);
m_add(TEMP2,TEMP4,TEMP5);
ms_mltadd(TEMP5,A1,(1.0 - 2.0 * delta),B1);

sm_mlt(delta,A1,TEMP6);
m_mlt(TEMP6,M3,TEMP7);
sm_mlt(delta,A3,TEMP8);
m_mlt(TEMP8,M4,TEMP9);
m_add(TEMP7,TEMP9,TEMP10);
ms_mltadd(TEMP10,A2,(1.0 - 2.0 * delta),B2);

sm_mlt(delta,A1,TEMP11);
m_mlt(TEMP11,M5,TEMP12);
sm_mlt(delta,A2,TEMP13);
m_mlt(TEMP13,M6,TEMP14);
m_add(TEMP12,TEMP14,TEMP15);
ms_mltadd(TEMP15,A3,(1.0 - 2.0 * delta),B3);


/*
 * Determinants needed to compute normal theory test
 */


for (i = 1; i <= r1; ++i)
   set_col(B12,i-1,get_col(B1,i-1,TEMPa));
for (i = 1; i <= r2; ++i)
   set_col(B12,i+r1-1,get_col(B2,i-1,TEMPa));

for (i = 1; i <= r2; ++i)
   set_col(B23,i-1,get_col(B2,i-1,TEMPb));
for (i = 1; i <= r3; ++i)
   set_col(B23,i+r2-1,get_col(B3,i-1,TEMPb));

for (i = 1; i <= r3; ++i)
```

```
    set_col(B31,i-1,get_col(B3,i-1,TEMPc));
for (i = 1; i <= r1; ++i)
    set_col(B31,i+r3-1,get_col(B1,i-1,TEMPc));

sub_mean(B12);
sub_mean(B23);
sub_mean(B31);
mtrm_mlt(B12,B12,TB12);
mtrm_mlt(B23,B23,TB23);
mtrm_mlt(B31,B31,TB31);
sub_matrix(TB12,0,0,r1-1,r1-1,S11);
sub_matrix(TB12,r1,r1,r12-1,r12-1,S22);
sub_matrix(TB12,0,r1,r1-1,r12-1,S12);
sub_matrix(TB12,r1,0,r12-1,r1-1,S21);
sub_matrix(TB23,r2,r2,r23-1,r23-1,S33);
sub_matrix(TB23,0,r2,r2-1,r23-1,S23);
sub_matrix(TB31,0,r3,r3-1,r31-1,S31);
sub_matrix(TB31,r3,0,r31-1,r3-1,S13);

for (i = 1; i <= r1; ++i)
    set_col(T1,i-1,get_col(S11,i-1,TEMPd));
for (i = 1; i <= r2; ++i)
    set_col(T1,i+r1-1,get_col(S12,i-1,TEMPd));

for (i = 1; i <= r1; ++i)
    set_col(T2,i-1,get_col(S21,i-1,TEMPe));
for (i = 1; i <= r2; ++i)
    set_col(T2,i+r1-1,get_col(S22,i-1,TEMPe));

for (i = 1; i <= r1; ++i)
    set_row(T3,i-1,get_row(T1,i-1,TEMPf));
for (i = 1; i <= r2; ++i)
    set_row(T3,i+r1-1,get_row(T2,i-1,TEMPf));

for (i = 1; i <= r1; ++i)
    set_row(T4,i-1,get_row(S13,i-1,TEMPg));
for (i = 1; i <= r2; ++i)
    set_row(T4,i+r1-1,get_row(S23,i-1,TEMPg));

m_inverse(S11,S11);
m_inverse(S22,S22);
m_inverse(S33,S33);
m_inverse(T3,T3);
m_mlt(S11,S12,TMP1);
m_mlt(S22,S23,TMP3);
```

```
m_mlt(S33,S31,TMP5);
m_mlt(T3,T4,TMP7);

mmtr_mlt(S22,S12,TMP2);
mmtr_mlt(S33,S23,TMP4);
mmtr_mlt(S11,S31,TMP6);
mmtr_mlt(S33,T4,TMP8);

m_mlt(TMP1,TMP2,SA);
m_mlt(TMP3,TMP4,SB);
m_mlt(TMP5,TMP6,SC);
m_mlt(TMP7,TMP8,SAB);

m_sub(IDA,SA,OUTA);
m_sub(IDB,SB,OUTB);
m_sub(IDC,SC,OUTC);
m_sub(IDAB,SAB,OUTAB);

/*
 * Compute statistics
 */


/* Wilks' lambda */

num1 = det(OUTA);
num2 = det(OUTB);
num3 = det(OUTC);
num4 = det(OUTAB);

L = -k*log(num1) - k*log(num4);

/* Wilks' lambda using an approximation */


La = -k1*log(num1) - k2*log(num2) - k3*log(num3);


/* Puri-Sen sign statistic */


PS = PuriSen(B1,B2) + PuriSen(B12,B3);


/* Puri-Sen sign statistic using an approximation */
```

```
PSa = PuriSen(B1,B2) + PuriSen(B2,B3) + PuriSen(B3,B1);

/* Quadrant statistics */

OjaMedian(B1,THETA1);
OjaMedian(B2,THETA2);
OjaMedian(B3,THETA3);

Q1 =  Quadrant1(B1,B2,THETA1,THETA2)
    + Quadrant1(B2,B3,THETA2,THETA3)
    + Quadrant1(B3,B1,THETA3,THETA1);
Q2 =  Quadrant2(B1,B2,THETA1,THETA2,5,2.0)
    + Quadrant2(B2,B3,THETA2,THETA3,5,2.0)
    + Quadrant2(B3,B1,THETA3,THETA1,5,2.0);

/* Reject? */


iL = (L > X);
iLa = (La > X);
iPS = (PS > X);
iPSa = (PSa > X);
iQ1 = (Q1 > X);
iQ2 = (Q2 > X);

/*
 * Update counters
 */

cL += iL;
cLa += iLa;
cPS += iPS;
cPSa += iPSa;
cQ1 += iQ1;
cQ2 += iQ2;

/*
 *  Print value of statistics
 */

if (option == 'p') {
  printf("\n");
  printf("%4d. ",reps - j + 1);
```

```
      printf("%6.2f",L);
      if (iL) printf("*"); else printf(" ");
      printf("%6.2f",La);
      if (iLa) printf("*"); else printf(" ");
      printf("%6.2f",PS);
      if (iPS) printf("*"); else printf(" ");
      printf("%6.2f",PSa);
      if (iPSa) printf("*"); else printf(" ");
      printf("%6.2f",Q1);
      if (iQ1) printf("*"); else printf(" ");
      printf("%6.2f",Q2);
      if (iQ2) printf("*");
    }
  }

  /*
   * Print empirical powers
   */

  if (option == 'p')
    printf("\n\nEmpirical Powers\n\n");

  printf("%4.2f ",delta);


  printf("%6.4f ",(double)cL/reps);
  printf("%6.4f ",(double)cLa/reps);
  printf("%6.4f ",(double)cPS/reps);
  printf("%6.4f ",(double)cPSa/reps);
  printf("%6.4f ",(double)cQ1/reps);
  printf("%6.4f\n",(double)cQ2/reps);

  /*
   * Save current generator and seeds
   */

  gscgn(0,&g);
  getsd(&seed1,&seed2);
  seeds = fopen(seedfile,"at");
  fprintf(seeds,"%ld %ld %ld\n",g,seed1,seed2);
  fclose(seeds);

  return 0;
}
```

```
/* exit with message */
int bye()
{
   printf(usage,cmd);
   exit(0);
}
```

.0.2 signrank.c

```
#include <stdio.h>
#include <math.h>
#include "general.h"

/*
 *  SignandRank returns the value of the sign and rank
 *  statistics based on interdirection
 */

double SignandRank(MAT *A1, MAT *A2,  MAT *A3, MAT *A12, MAT
    *A23, MAT *A31,
     MAT *ATOT,  VEC *THETA, double *QXYZ1, double *QXYZ2)
{
   double Q1a, Q2a, Q3a, Q1b, Q2b, Q3b, Q12a, Q23a, Q31a;
   double Q12b, Q23b, Q31b, QXYa, QXYb, QYZa, QYZb, QZXa, QZXb;
   double phix1,phix2,phiy1,phiy2,phiz1,phiz2,phixy1,phixy2;
   double phiyz1,phiyz2,phizx1,phizx2,xx,yy,zz;
   double phixd,phiyd,phizd,phix,phiy,phiz,phixy,phiyz,phizx,d1,d2,d3;
   double temp1, temp2, temp3, temp12, temp23, temp31;
   double adj1, adj2, adj3, adj12, adj23, adj31;
   int n1, n2, n3, n12, n23, n31, n, adj_n, r, adj_n1, adj_n2, adj_n3 ;
   int nhyps1, nhyps2, nhyps3, nhyps12, nhyps23, nhyps31;
   int i, j, z1index, z2index, z3index, z12index, z23index, z31index,
   MAT *T1, *T2, *T3,  *T12, *T23, *T31;
   MAT *COUNTS1, *COUNTS2, *COUNTS3, *COUNTS12, *COUNTS23, *COUNTS31;
   MAT *OUT1, *OUT2, *OUT3, *OUT4, *OUT5,*SIGMA;
   VEC  *TEMPe, *TEMPf, *TEMPg;
   VEC *TMP1, *TMP2, *TMP3, *DISTANCE;

   n1 = A1->m;
   n2 = A2->m;
   n3 = A3->m;
   n12 = n1 + n2;
   n23 = n2 + n3;
```

```
n31 = n3 + n1;
n = n1 + n2 + n3;
r = A1->n;
T1 = get_mat(n1,r);
T2 = get_mat(n2,r);
T3 = get_mat(n3,r);
T12 = get_mat(n12,r);
T23 = get_mat(n23,r);
T31 = get_mat(n31,r);
OUT1 = get_mat(r,r);
OUT2 = get_mat(r,r);
OUT3 = get_mat(r,r);
OUT4 = get_mat(r,r);
OUT5 = get_mat(r,r);
SIGMA = get_mat(r,r);
TMP1 = get_vec(r);
TMP2 = get_vec(r);
COUNTS1 = get_mat(n1,n1);
COUNTS2 = get_mat(n2,n2);
COUNTS3 = get_mat(n3,n3);
COUNTS12 = get_mat(n12,n12);
COUNTS23 = get_mat(n23,n23);
COUNTS31 = get_mat(n31,n31);
TEMPe = get_vec(r);
TEMPf = get_vec(r);
TEMPg = get_vec(r);
TMP3 = get_vec(r);
DISTANCE = get_vec(n);

cp_mat(A1,T1);
cp_mat(A2,T2);
cp_mat(A3,T3);
cp_mat(A12,T12);
cp_mat(A23,T23);
cp_mat(A31,T31);

/*
 * This is a patch to adjust the statistic if one of the
 * points happens to be equal to the location estimate
 */

z1index = Center(T1,THETA);
z2index = Center(T2,THETA);
z3index = Center(T3,THETA);
z12index = Center(T12,THETA);
```

```
z23index = Center(T23,THETA);
z31index = Center(T31,THETA);

mtrm_mlt(T1,T1,OUT1);
mtrm_mlt(T2,T2,OUT2);
mtrm_mlt(T3,T3,OUT3);
m_add(OUT1,OUT2,OUT4);
m_add(OUT4,OUT3,OUT5);
sm_mlt(1.0 / n,OUT5,SIGMA);

m_inverse(SIGMA,SIGMA) ;

for (i = 0; i < n1; ++i) {
    get_row(T1,i,TEMPe);
    vm_mlt(SIGMA,TEMPe,TMP1);
    d1 =  in_prod(TMP1,TEMPe);
    DISTANCE->ve[i] = d1;
 }
for (i = 0; i < n2; ++i) {
    get_row(T2,i,TEMPf);
    vm_mlt(SIGMA,TEMPf,TMP2);
    d2 =  in_prod(TMP2,TEMPf);
    DISTANCE->ve[n1+i] = d2;
 }
for (i = 0; i < n3; ++i) {
    get_row(T3,i,TEMPg);
    vm_mlt(SIGMA,TEMPg,TMP3);
    d3 =  in_prod(TMP3,TEMPg);
    DISTANCE->ve[n12+i] = d3;
 }

rank(DISTANCE,DISTANCE);

adj_n1 = n1 - (z1index > 0);
adj_n2 = n2 - (z2index > 0);
adj_n3 = n3 - (z3index > 0);
adj_n = n - (z1index > 0) - (z2index > 0)  - (z3index > 0)
 +  (z1index > 0 && z2index > 0) + (z2index > 0 && z3index > 0)
 + (z1index > 0 && z3index > 0) - (z1index > 0 && z2index > 0 &&
   z3index > 0);


nhyps1 = InterdirectionCounts(T1,COUNTS1);
nhyps2 = InterdirectionCounts(T2,COUNTS2);
nhyps3 = InterdirectionCounts(T3,COUNTS3);
```

```
nhyps12 = InterdirectionCounts(T12,COUNTS12);
nhyps23 = InterdirectionCounts(T23,COUNTS23);
nhyps31 = InterdirectionCounts(T31,COUNTS31);
adj1 = (double)((n1 − r + 1) ∗ (n1 − r))/(n1 ∗ (n1 − 1));
adj2 = (double)((n2 − r + 1) ∗ (n2 − r))/(n2 ∗ (n2 − 1));
adj3 = (double)((n3 − r + 1) ∗ (n3 − r))/(n3 ∗ (n3 − 1));
adj12 = (double)((n12 − r + 1) ∗ (n12 − r))/(n12 ∗ (n12 − 1));
adj23 = (double)((n23 − r + 1) ∗ (n23 − r))/(n23 ∗ (n23 − 1));
adj31 = (double)((n31 − r + 1) ∗ (n31 − r))/(n31 ∗ (n31 − 1));

Q1a = 0.0;  Q1b = 0.0;
Q2a = 0.0;  Q2b = 0.0;
Q3a = 0.0;  Q3b = 0.0;
Q12a = 0.0;  Q12b = 0.0;
Q23a = 0.0;  Q23b = 0.0;
Q31a = 0.0;  Q31b = 0.0;
phixd = 0.0; phiyd = 0.0; phizd = 0.0;

for (i = 1; i ≤ n1 − 1; ++i) {
   phix1 = DISTANCE→ve[i−1];
   phixd = phixd + phix1 ∗ phix1/(n ∗ n);
   for (j = i + 1; j ≤ n1; ++j) {
      temp1 = m_entry(COUNTS1,i−1,j−1)/(adj1 ∗ nhyps1);
      phix2 = DISTANCE→ve[j−1];
      phix = phix1 ∗ phix2 /(n∗n);
      Q1a += cos(PI ∗ temp1);
      Q1b += cos(PI ∗ temp1) ∗ phix;
   }
}
xx = DISTANCE→ve[n1 − 1] ;
phixd = phixd  + xx ∗ xx /(n ∗ n);
Q1a = 2.0 ∗ Q1a / (adj_n1 ∗ adj_n1) + 1.0 / adj_n1;
Q1b = 2.0 ∗ Q1b / (adj_n1 ∗ adj_n1) + phixd /(adj_n1 ∗ adj_n1);

for (i = 1; i ≤ n2 − 1; ++i) {
   phiy1 = DISTANCE→ve[n1+i−1];
   phiyd = phiyd + phiy1 ∗ phiy1/(n ∗ n);
   for (j = i + 1; j ≤ n2; ++j) {
      temp2 = m_entry(COUNTS2,i−1,j−1)/(adj2 ∗ nhyps2);
      phiy2 = DISTANCE→ve[n1+j−1];
      phiy = phiy1 ∗ phiy2 / (n∗n);
      Q2a += cos(PI ∗ temp2);
      Q2b += cos(PI ∗ temp2) ∗ phiy;
   }
}
```

```
 yy = DISTANCE→ve[n12 − 1] ;
phiyd = phiyd  + yy ∗ yy /(n ∗ n);
Q2a = 2.0 ∗ Q2a / (adj_n2 ∗ adj_n2) + 1.0 / adj_n2;
Q2b = 2.0 ∗ Q2b / (adj_n2 ∗ adj_n2) + phiyd /(adj_n2 ∗ adj_n2);

for (i = 1; i ≤ n3  − 1; ++i) {
   phiz1 = DISTANCE→ve[n12+i−1];
   phizd = phizd + phiz1 ∗ phiz1/(n ∗ n);
   for (j = i + 1; j ≤ n3; ++j) {
      temp3 = m_entry(COUNTS3,i−1,j−1)/(adj3 ∗ nhyps3);
      phiz2 = DISTANCE→ve[n12+j−1];
      phiz = phiz1 ∗ phiz2 / (n∗n);
      Q3a += cos(PI ∗ temp3);
      Q3b += cos(PI ∗ temp3) ∗ phiz ;
   }
}
zz = DISTANCE→ve[n − 1] ;
phizd = phizd  + zz ∗ zz /(n ∗ n);
Q3a = 2.0 ∗ Q3a / (adj_n3 ∗ adj_n3) + 1.0 / adj_n3;
Q3b = 2.0 ∗ Q3b / (adj_n3 ∗ adj_n3)+ phizd /(adj_n3 ∗ adj_n3);

for (i = 1; i ≤ n1; ++i) {
   for (j = n1+1; j ≤ n12; ++j) {
      temp12 = m_entry(COUNTS12,i−1,j−1)/(adj12 ∗ nhyps12);
      phixy1 = DISTANCE→ve[i−1];
      phixy2 = DISTANCE→ve[j−1];
      phixy = phixy1 ∗ phixy2 / (n ∗ n);
      Q12a += cos(PI ∗ temp12);
      Q12b += cos(PI ∗ temp12) ∗ phixy;
   }
}
Q12a =  2.0 ∗ Q12a / (adj_n1 ∗ adj_n2);
Q12b =  2.0 ∗ Q12b / (adj_n1 ∗ adj_n2);

for (i = 1; i ≤ n2; ++i) {
   for (j = n2+1; j ≤ n23; ++j) {
      temp23 = m_entry(COUNTS23,i−1,j−1)/(adj23 ∗ nhyps23);
      phiyz1 = DISTANCE→ve[n1+i−1];
      phiyz2 = DISTANCE→ve[n1+j−1];
      phiyz = phiyz1 ∗ phiyz2 / (n ∗ n);
      Q23a += cos(PI ∗ temp23);
      Q23b += cos(PI ∗ temp23) ∗ phiyz;
   }
}
Q23a =  2.0 ∗ Q23a / (adj_n2 ∗ adj_n3);
```

```
    Q23b =  2.0 * Q23b / (adj_n2 * adj_n3);

    for (i = 1; i <= n3; ++i) {
        for (j = n3+1; j <= n31; ++j) {
            temp31 = m_entry(COUNTS31,i-1,j-1)/(adj31 * nhyps31);
            phizx1 = DISTANCE->ve[n12+i-1];
            phizx2 = DISTANCE->ve[j-n3-1];
            phizx = phizx1 * phizx2 / (n * n);
            Q31a += cos(PI * temp31);
            Q31b += cos(PI * temp31) *  phizx;
        }
    }
    Q31a = 2.0 * Q31a / (adj_n3 * adj_n1);
    Q31b = 2.0 * Q31b / (adj_n3 * adj_n1);

    QXYa = adj_n1 * adj_n2 * r *(Q1a + Q2a - Q12a) / adj_n ;
    QYZa = adj_n2 * adj_n3 * r *(Q2a + Q3a - Q23a) / adj_n ;
    QZXa = adj_n3 * adj_n1 * r *(Q3a + Q1a - Q31a) / adj_n ;
    QXYb = adj_n1 * adj_n2 * r *(Q1b + Q2b - Q12b) * 3.0 / adj_n ;
    QYZb = adj_n2 * adj_n3 * r *(Q2b + Q3b - Q23b) * 3.0 / adj_n ;
    QZXb = adj_n3 * adj_n1 * r *(Q3b + Q1b - Q31b) * 3.0 / adj_n ;

    freemat(COUNTS1);
    freemat(COUNTS2);
    freemat(COUNTS3);
    freemat(COUNTS12);
    freemat(COUNTS23);
    freemat(COUNTS31);
    freemat(T1);
    freemat(T2);
    freemat(T3);
    freemat(T12);
    freemat(T23);
    freemat(T31);
    freemat(OUT1);
    freemat(OUT2);
    freemat(OUT3);

   *QXYZ1 = QXYa + QYZa + QZXa;
   *QXYZ2 = QXYb + QYZb + QZXb;
}
```

.0.3 location.c

```
#include <stdio.h>
#include <math.h>
#include "ranlib.h"
#include "general.h"

#define NOMINAL 0.05

/*
 * This is the main routine which does the repetitions
 * and counts the number of times each of the competing
 * statistics falls in the rejection region.
 */

char *usage = "Usage: %s [p or n] dim N reps delta el1 el2 el3 nu seedfile\n";
char *cmd;

main(int argc, char **argv)
{

  double  delta, el1, el2, el3, nu, c;
  int i, j, jj, dim, N2, N3, r, N, reps;
  double  L, LH, LHF, PS2, QSIGN, QRANK, HO, QXYZ1, QXYZ2;
  double X, d1, d2, d3, const1, const2, F1;
  MAT *A1, *A2, *A3, *B2, *B3, *A12, *A23, *A31;
  MAT *ATOT, *SIGMA, *MTMP1, *MTMP2;
  MAT *MT1, *MT2, *MT3, *MT4, *MT5, *OUT1,*OUT2, *OUT3, *OUT4, *OUT5;
  MAT *OUT6, *OUT7, *OUT8, *OUT9, *OUT10, *OUT11, *OUT12;
  VEC *TEMPa, *TEMPb, *TEMPc, *TEMPd, *THETA, *TEMP1, *TEMP2, *TEMP3;
  VEC *TEMP4, *TEMP5, *TEMP6, *TMP;
  VEC *TX, *TY, *TZ, *TXYZ, *T1, *T2, *T3, *T4, *UNITY, *ALL;
  int  cLHF, cPS2, cQSIGN, cQRANK, cHO;
  int  iLHF, iPS2, iQSIGN, iQRANK, iHO, rnum, cnum;

  char option;

  long seed1, seed2, g;
  char seedfile[80];
  FILE *seeds;

  cmd = *argv;

  if (argc < 11) bye();
```

```
if (sscanf(argv[1],"%s",&option) ≠ 1) bye();

if (sscanf(argv[2],"%u",&dim) ≠ 1) bye();
if (dim < 1 || dim > MAX_DIM) {
  printf("Invalid value of dim\n");
  exit(0);
}

if (sscanf(argv[3],"%u",&N) ≠ 1) bye();
if (N < 2 * dim + 1 || N > MAX_N) {
  printf("Invalid value of N\n");
  exit(0);
}

if (sscanf(argv[4],"%u",&reps) ≠ 1) bye();
if (reps < 1) {
  printf("Invalid value of reps\n");
  exit(0);
}

if (sscanf(argv[5],"%lf",&delta) ≠ 1) bye();
if (delta < 0.0 || delta > 1.0) {
  printf("Invalid value of delta1\n");
  exit(0);
}

if (sscanf(argv[6],"%lf",&el1) ≠ 1) bye();

if (sscanf(argv[7],"%lf",&el2) ≠ 1) bye();

if (sscanf(argv[8],"%lf",&el3) ≠ 1) bye();


if (sscanf(argv[9],"%lf",&nu) ≠ 1) bye();
if (nu ≤ 0.0) {
  printf("Invalid value of nu\n");
  exit(0);
}

if (sscanf(argv[10],"%s",&seedfile) ≠ 1) bye();

/*
 * Set generator number and current seeds
 */
```

```
seeds = fopen(seedfile,"rt");
while(fscanf(seeds,"%ld %ld %ld",&g,&seed1,&seed2) ≠ EOF);
fclose(seeds);
gscgn(1,&g);
setall(seed1,seed2);

 r = dim;
 N2  = N + N;
 N3 = N + N + N;
 A1 = get_mat(N,r);
 A2 = get_mat(N,r);
 A3 = get_mat(N,r);
 B2 = get_mat(N,r);
 B3 = get_mat(N,r);
 MTMP1 = get_mat(N,r);
 MTMP2 = get_mat(N,r);
 A12 = get_mat(N2,r);
 A23 = get_mat(N2,r);
 A31 = get_mat(N2,r);
 ATOT = get_mat(N3,r);
MT1 = get_mat(N,r);
MT2 =get_mat(N,r);
MT3 = get_mat(N,r);
MT4 = get_mat(r,r);
MT5 = get_mat(r,r);
OUT1 = get_mat(N,r);
OUT2 = get_mat(N,r);
OUT3 = get_mat(N,r);
OUT4 = get_mat(r,r);
OUT5 = get_mat(r,r);
OUT6 = get_mat(r,r);
OUT7 = get_mat(r,r);
OUT8 = get_mat(r,r);
OUT9 = get_mat(r,r);
OUT10 =get_mat(r,r);
OUT11 = get_mat(r,r);
OUT12 = get_mat(r,r);
SIGMA = get_mat(r,r);
UNITY = get_vec(N);
ALL = get_vec(N3);
TX = get_vec(r);
TY = get_vec(r);
TZ = get_vec(r);
TXYZ = get_vec(r);
```

```
T1 = get_vec(r);
T2 = get_vec(r);
T3 = get_vec(r);
T4 = get_vec(r);
TEMP1 = get_vec(r);
TEMP2 = get_vec(r);
TEMP3= get_vec(r);
TEMP4 = get_vec(r);
TEMP5= get_vec(r);
TEMP6= get_vec(r);
TEMPa = get_vec(r);
TEMPb = get_vec(r);
TEMPc = get_vec(r);
TEMPd = get_vec(r);
THETA = get_vec(r);
TMP = get_vec(r);
ones_mat(MTMP1);
ones_vec(UNITY);
ones_vec(ALL);


X = critchi(NOMINAL,2*r);
F1 = critf(NOMINAL,6,78);

cLHF =  cQSIGN = cQRANK = cPS2 = cHO = 0;


if (option == 'p') {
  printf("\n\n    LHF  PS2   QSIGN   QRANK   HO \n");
}

for(j = reps; j > 0; --j) {

  /*
   * Generate the initial random vectors
   */

  rand_dat(A1,nu);
  rand_dat(B2,nu);
  rand_dat(B3,nu);

/*
 * Generate the random vectors with different mean
 */
```

```
TMP->ve[0] = el1; TMP->ve[1] = el2; TMP->ve[2] = el3;

ms_mltadd(B2,MTMP1,delta,A2);

for(i=0;i<N;++i)
  set_row(MTMP2,i,TMP);

m_add(B3,MTMP2,A3);


for ( i = 1; i <= N; ++i)
    set_row(A12, i-1,get_row(A1,i-1,TEMPa));
for ( i = 1; i <= N; ++i)
    set_row(A12, i+N-1,get_row(A2,i-1,TEMPa));

for ( i = 1; i <= N; ++i)
    set_row(A23, i-1,get_row(A2,i-1,TEMPb));
for ( i = 1; i <= N; ++i)
    set_row(A23, i+N-1,get_row(A3,i-1,TEMPb));

for ( i = 1; i <= N; ++i)
    set_row(A31, i-1,get_row(A3,i-1,TEMPc));
for ( i = 1; i <= N; ++i)
    set_row(A31, i+N-1,get_row(A1,i-1,TEMPc));

for ( i = 1; i <= N2; ++i)
    set_row(ATOT, i-1,get_row(A12,i-1,TEMPd));
for ( i = 1; i <= N; ++i)
    set_row(ATOT, i+N2-1,get_row(A3,i-1,TEMPd));

/*
 * Compute statistics
 */


rnum = ATOT->m;
cnum = ATOT->n;

/* Lawley - Hotelling generalized statistic */

vm_mlt(A1,UNITY,TX);
vm_mlt(A2,UNITY,TY);
vm_mlt(A3,UNITY,TZ);
vm_mlt(ATOT,ALL,TXYZ);
```

```
const1 = 1.0 / N ;
const2 = 1.0 / N3 ;

 sv_mlt(const1,TX,T1);
 sv_mlt(const1,TY,T2);
 sv_mlt(const1,TZ,T3);
 sv_mlt(const2,TXYZ,T4);

 for(i=0; i < N; ++i) {
  set_row(MT1,i,T1);
  set_row(MT2,i,T2);
  set_row(MT3,i,T3);
}

 m_sub(A1,MT1,OUT1);
 m_sub(A2,MT2,OUT2);
 m_sub(A3,MT3,OUT3);
 mtrm_mlt(OUT1,OUT1,OUT4);
 mtrm_mlt(OUT2,OUT2,OUT5);
 mtrm_mlt(OUT3,OUT3,OUT6);
 m_add(OUT4,OUT5,OUT7);
 m_add(OUT6,OUT7,OUT8);

for(i=0; i < r; ++i)
  set_col(MT4,i,T4);

for(i=0; i < r; ++i){
      MT5->me[i][0] = T1->ve[i];
      MT5->me[i][1] = T2->ve[i];
      MT5->me[i][2] = T3->ve[i];
 }

 m_sub(MT5,MT4,OUT9);
 mmtr_mlt(OUT9,OUT9,OUT10);
 sm_mlt(1/const1,OUT10,OUT11);
 m_add(OUT8,OUT11,OUT12);

 c = 3.0 * N - 3.0;
 sm_mlt(1/c,OUT8,SIGMA);
 m_inverse(SIGMA,SIGMA);

 v_sub(T1,T4,TEMP1);
 vm_mlt(SIGMA,TEMP1,TEMP2);
 d1 = in_prod(TEMP2,TEMP1);
```

```
v_sub(T2,T4,TEMP3);
vm_mlt(SIGMA,TEMP3,TEMP4);
d2 = in_prod(TEMP4,TEMP3);
v_sub(T3,T4,TEMP5);
vm_mlt(SIGMA,TEMP5,TEMP6);
d3 = in_prod(TEMP6,TEMP5);

LH = N * (d1 + d2 + d3);
LHF = LH * 13.0 / 84.0;

/* Puri-Sen sign statistic */


PS2= PuriSen2(A12,A3);

/* SIGN and RANK  statistics */

OjaMedian(ATOT,THETA);


SignandRank(A1,A2,A3,A12,A23,A31,ATOT,THETA,&QXYZ1,&QXYZ2);

 QSIGN = QXYZ1;
 QRANK = QXYZ2;

/* Hettmansperger and Oja statistics  */

 HO =  hettoja(ATOT,THETA);


/* Reject? */

iLHF = (LHF > F1);
iPS2 = (PS2 > X);
iQSIGN = (QSIGN  > X);
iQRANK = (QRANK > X);
iHO = (HO  > X);

/*
 * Update counters
 */

cLHF += iLHF;
cPS2 += iPS2;
cQSIGN += iQSIGN;
```

```
    cQRANK += iQRANK;
    cHO += iHO;

    /*
     *  Print value of statistics
     */

    if (option == 'p') {
      printf("\n");
      printf("%4d. ",reps - j + 1);

      printf("%6.2f",LHF);
      if (iLHF) printf("*"); else printf(" ");
      printf("%6.2f",PS2);
      if (iPS2) printf("*"); else printf(" ");
      printf("%6.2f",QSIGN);
      if (iQSIGN) printf("*"); else printf(" ");
      printf("%6.2f",QRANK);
      if (iQRANK) printf("*"); else printf(" ");
      printf("%6.2f",HO);
      if (iHO) printf("*");
    }
  }

  /*
   * Print empirical powers
   */

  if (option == 'p')
    printf("\n\nEmpirical Powers\n\n");

  printf("%4.2f ",delta);
  printf("%4.2f ",el1);
  printf("%4.2f ",el2);
  printf("%4.2f ",el3);

  printf("%6.4f ",(double)cLHF/reps);
  printf("%6.4f ",(double)cPS2/reps);
  printf("%6.4f ",(double)cQSIGN/reps);
  printf("%6.4f",(double)cQRANK/reps);
  printf("%6.4f\n",(double)cHO/reps);

  /*
   * Save current generator and seeds
   */
```

```
  gscgn(0,&g);
  getsd(&seed1,&seed2);
  seeds = fopen(seedfile,"at");
  fprintf(seeds,"%ld %ld %ld\n",g,seed1,seed2);
  fclose(seeds);

  return 0;
}

/* exit with message */
int bye()
{
   printf(usage,cmd);
   exit(0);
}
```

REFERENCES

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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